# TORIC ORBIFOLDS ASSOCIATED TO CARTAN MATRICES

MARK BLUME

ABSTRACT. We investigate moduli stacks of pointed chains of projective lines related to the Losev-Manin moduli spaces and show that these moduli stacks coincide with certain toric stacks which can be described in terms of the Cartan matrices of root systems of type $A$. We also consider variants of these stacks related to root systems of type $B$ and $C$.

## CONTENTS



## INTRODUCTION

The Losev-Manin moduli spaces $\overline{L}_n$, introduced in [LM00], parametrise isomorphism classes of stable $n$-pointed chains of projective lines. The space $\overline{L}_n$ forms a compactification of the torus $(\mathbb{G}_m)^n/\mathbb{G}_m$ that parametrises $n$ points $s_1, \ldots, s_n$ in $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$ up to automorphisms of $\mathbb{P}^1$ fixing the two points $0, \infty$. It is a smooth projective toric variety isomorphic to the toric variety $X(A_{n-1})$ associated with the root system $A_{n-1}$, see [BB11a].

---

Supported by DFG-Schwerpunkt 1388 Darstellungstheorie.

In the present paper we are concerned with a variant of the Losev-Manin moduli spaces which arises as a compactification of the moduli space of $n$ non-distinguishable points in $\mathbb{P}^1 \setminus \{0, \infty\}$, or equivalently, finite subschemes of degree $n$ in $\mathbb{P}^1 \setminus \{0, \infty\}$. Isomorphism classes of such subschemes correspond to polynomials of the form $y^n + a_{n-1}y^{n-1} + \ldots + a_1 y + 1$ up to multiplication of the variable $y$ by an $n$-th root of unity. The torus $(\mathbb{G}_m)^{n-1}$, parametrising polynomials with non-zero coefficients $a_1, \ldots, a_{n-1}$, is compactified by the moduli stack of chains of projective lines with finite subschemes of degree $n$. On the boundary both the coefficients of the polynomials may become zero and the curve may become a reducible chain of projective lines. The category of these pointed curves, which we call degree-$n$-pointed chains of projective lines, forms an orbifold $\overline{\mathcal{L}}_n$.

The orbifold $\overline{\mathcal{L}}_n$ is related to the Losev-Manin moduli space $\overline{L}_n$ by an $S_n$-equivariant morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$, $\overline{L}_n$ with the operation of the symmetric group $S_n$ that permutes the $n$ sections and $\overline{\mathcal{L}}_n$ with trivial operation, which is given by mapping an $n$-pointed chain of projective lines to the corresponding degree-$n$-pointed chain by forgetting the labels of the sections. The moduli stack $\overline{\mathcal{L}}_n$ is defined such that the morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$ is closely related to morphisms of the form $C_0^n \to C_0^n/S_n = C_0^{(n)} = \mathrm{Div}^n_{C_0/Y}$ from the $n$-fold product over $Y$ to the scheme of relative effective Cartier divisors of degree $n$ for $C_0 \to Y$ a relative smooth curve over $Y$, here a chain of projective lines over $Y$ without the poles of the components of the fibres. Therefore the morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$ inherits properties like being faithfully flat and finite of degree $n!$ and being ramified exactly in the points corresponding to curves with coinciding marked points, see proposition 1.5. The stack $\overline{\mathcal{L}}_n$ differs from the quotient stack $[\overline{L}_n/S_n]$, it has the same points but different automorphism groups. The coarse moduli space of $\overline{\mathcal{L}}_n$ coincides with the quotient $\overline{L}_n/S_n$.

A main result of this paper, theorem 3.1, is an explicit description of the structure of the stacks $\overline{\mathcal{L}}_n$: we show that $\overline{\mathcal{L}}_n$ is a toric orbifold and we determine the associated combinatorial data.

Toric Deligne-Mumford stacks over fields of characteristic 0 were introduced in [BCS05] and constructed from combinatorial data called (simplicial) stacky fans, consisting of a simplicial fan and some extra data, as quotient stacks $[U/T]$ of an open subscheme $U$ of some affine space by a diagonalisable group scheme $G$, generalising the quotient construction of a smooth toric variety described in [Co95a]. Over more general base schemes in the same way these data give rise to toric stacks which are not necessarily Deligne-Mumford stacks but tame stacks in the sense of [AOV08]. As our moduli problem results in stacks which are orbifolds, in this paper we are mainly concerned with toric orbifolds, i.e. toric tame stacks with trivial generic stabiliser. We will mainly work with toric orbifolds over the integers, considering the fact that our moduli problem is naturally defined over the integers.

It turns out that the moduli stacks $\overline{\mathcal{L}}_n$ can be described in terms of the Cartan matrices of root systems of type $A$, more precisely, $\overline{\mathcal{L}}_n$ is isomorphic to the toric orbifold $\mathcal{Y}(A_{n-1})$ which corresponds to the stacky fan $\boldsymbol{\Upsilon}(A_{n-1})$ defined in section 2 using the Cartan matrix of the root system $A_{n-1}$. For the proof of the isomorphism $\overline{\mathcal{L}}_n \cong \mathcal{Y}(A_{n-1})$ we make use of a generalisation of the description of the functor of toric varieties [Co95b] for toric stacks, which allows to characterise $\mathcal{Y}(A_{n-1})$ as a stack $\mathcal{C}_{\boldsymbol{\Upsilon}(A_{n-1})}$ of $\boldsymbol{\Upsilon}(A_{n-1})$-collections, i.e. collections of pairs of a line bundle with a section and additional data.

We also characterise the morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$, determined by forgetting the labels of the $n$ sections, in terms of the combinatorial data by specifying the $\boldsymbol{\Upsilon}(A_{n-1})$-collection on $X(A_{n-1}) \cong \overline{L}_n$ corresponding to this morphism, see theorem 4.14. In doing this, in section 4 we compare the description of the functor of the toric varieties $X(A_{n-1})$ associated with root systems of type $A$ after Cox [Co95b] in terms of $\Sigma(A_{n-1})$-collections to two other descriptions: the description of [BB11a] in terms of $A_{n-1}$-data and a new description involving $S_n$-invariant line bundles on $X(A_{n-1})$. Both of these are related to Minkowski sum decompositions of the permutohedron: the first is a decomposition into line segments and the second corresponds to an embedding $X(A_{n-1}) \to \prod_{j=1}^{n-1} \mathbb{P}^{\binom{n}{j}-1}$ and expresses the permutohedron as sum of $S_n$-symmetric polytopes.

Generalisations of the Losev-Manin moduli spaces were investigated in [BB11b]. We considered $(2n+1)$-pointed and $2n$-pointed chains of projective lines with involution and showed that the moduli spaces $\overline{L}_n^{0,\pm}$ and $\overline{L}_n^{\pm}$ of these objects coincide with the toric varieties $X(B_n)$ and $X(C_n)$ associated with the root systems $B_n$ and $C_n$, see [BB11b, Thm. 4.1 and 6.15].

In the present setting it makes sense to investigate similar generalisations of the moduli stacks $\overline{\mathcal{L}}_n$ and to relate these to the toric orbifolds $\mathcal{Y}(R)$ for root systems $R$ belonging to other classical families as well as to the moduli spaces $\overline{L}_n^{0,\pm} \cong X(B_n)$ and $\overline{L}_n^{\pm} \cong X(C_n)$. In section 5 we consider moduli stacks of stable degree-$(2n+1)$-pointed and degree-$2n$-pointed chains of projective lines with involution, $\overline{\mathcal{L}}_n^{0,\pm}$ and $\overline{\mathcal{L}}_n^{\pm}$. We show that $\overline{\mathcal{L}}_n^{\pm}$ has a main component $\overline{\mathcal{L}}_{n,+}^{\pm}$ isomorphic to $\mathcal{Y}(C_n)$ and that $\overline{\mathcal{L}}_n^{0,\pm}$ is isomorphic to $\mathcal{Y}(B_n)^{\mathrm{can}}$, the canonical stack associated to $\mathcal{Y}(B_n)$ (see [FMN10]). We have morphisms $\overline{L}_n^{0,\pm} \to \overline{\mathcal{L}}_n^{0,\pm}$ and $\overline{L}_n^{\pm} \to \overline{\mathcal{L}}_{n,+}^{\pm}$, defined by forgetting the labels of the sections, which are equivariant under the Weyl group.

*Acknowledgements.* Thanks to Victor Batyrev.

## 1. Moduli stacks of degree-$n$-pointed chains

We define moduli stacks of stable degree-$n$-pointed chains of projective lines. Compared to the Losev-Manin moduli spaces considered in [LM00], [BB11a], we replace the $n$ marked points $s_1, \dots, s_n$ of an $n$-pointed chain of projective lines by a finite closed subscheme $S$ of degree $n$.

**Definition 1.1.** A stable degree-$n$-pointed chain of projective lines over an algebraically closed field $K$ is a tuple $(C, s_-, s_+, S)$, where $C$ is a chain of projective lines over $K$ with two distinct closed points $s_-, s_+$ on the outer components such that on each component the number of intersection points together with $s_-, s_+$ adds up to 2 (cf. [BB11a, Def. 3.1]), and $S \subset C$ a finite closed subscheme of degree $n$ that does neither meet the intersection points of components nor $s_-, s_+$, but that does meet every component of $C$. We define the category $\overline{\mathcal{L}}_n$ of stable degree-$n$-pointed chains of projective lines over the category of schemes. The objects over a scheme $Y$ are stable degree-$n$-pointed chains of projective lines over $Y$, i.e. tupels $\mathscr{C} = (C \to Y, s_-, s_+, S)$, where $C \to Y$ is a locally finitely presented, flat, proper morphism of schemes, $s_-, s_+ \colon Y \to C$ are sections and $S \subset C$ is a subscheme finite flat over $Y$, such that the geometric fibres are stable degree-$n$-pointed chains of projective lines. We have the natural notion of isomorphism of degree-$n$-pointed chains of projective lines over the same scheme $Y$ and of pullback of an object over a scheme $Y$ with respect to a morphism $f \colon Y' \to Y$. A morphism in $\overline{\mathcal{L}}_n$ over a morphism $f \colon Y' \to Y$ is a cartesian diagram of stable degree-$n$-pointed chains of projective lines over $f$.

**Remark 1.2.** (1) For a chain of projective lines $(C, s_-, s_+)$ over a field $K$ any component is isomorphic to $\mathbb{P}^1_K$ since it contains a point with residue field $K$.
(2) As the morphisms $C \to Y$ are locally finitely presented, by [EGA, IV, (8.9.1)] we can use some results which originally require some noetherian hypothesis.

**Remark 1.3.** The automorphism group of a chain of projective lines $(C, s_-, s_+)$ of length $l$ over a field $K$ is a torus $(\mathbb{G}_m)^l_K$. A stable degree-$n$-pointed chain of projective lines $(C, s_-, s_+, S)$ of length $l$ over $K$ has a finite automorphism group scheme which is a subgroup scheme of $(\mathbb{G}_m)^l_K$. There are objects $(C, s_-, s_+, S)$ having nontrivial automorphisms: consider for example a projective line $\mathbb{P}^1_K$ with homogeneous coordinates $z_0, z_1$, two poles $s_- = (1:0)$, $s_+ = (0:1)$ and a subscheme $S$ of degree $k$ given by the equation $z_0^k - z_1^k = 0$; in this example we have an automorphism group scheme isomorphic to $\mu_k$.

**Proposition 1.4.** *The category $\overline{\mathcal{L}}_n$ is a category fibred in groupoids over the category of schemes. It forms a stack over the fpqc site of schemes with representable, finite diagonal. Over fields of characteristic* 0 *the diagonal is unramified.*

*Proof.* The category $\overline{\mathcal{L}}_n$ together with the natural functor to the category of schemes is a fibred category, the cartesian arrows being cartesian diagrams of degree-$n$-pointed chains, and moreover the fibres $\overline{\mathcal{L}}_n(Y)$ over schemes $Y$ form a groupoid.

The fibred category $\overline{\mathcal{L}}_n$ is a prestack in the fpqc topology, i.e. descent data for morphisms are effective, see for example [Vi05, Prop. 4.31]. To show that $\overline{\mathcal{L}}_n$ is a

stack, it remains to show that descent data for objects are effective. Let $(\pi\colon C \to Y, s_-, s_+, S)$ be a stable degree-$n$-pointed chain of projective lines over a scheme $Y$. The subscheme $S \subset C$ is an effective Cartier divisor in $C$ because this is true on the fibres, see [Kl05, Lemma 9.3.4], and so its ideal sheaf $\mathscr{I} \subset \mathcal{O}_C$ is a line bundle. The line bundle $\mathcal{O}_C(S) = \mathscr{I}^{-1}$ is relatively ample with respect to $\pi$ since it is ample on the fibres, see [EGA, III, (4.7.1)], [EGA, IV, (9.6.5)]. In fact, $\mathcal{O}_C(S)$ defines a closed embedding in the projective bundle $\mathbb{P}_Y(\pi_*\mathcal{O}_C(S))$, see proposition 3.4. Given a morphism $F\colon (C' \to Y', s'_-, s'_+, S') \to (C \to Y, s_-, s_+, S)$ of two degree-$n$-pointed chains over a morphism $f\colon Y' \to Y$ forming a cartesian diagram, we have a natural isomorphism $F^*\mathcal{O}_C(S) \cong \mathcal{O}_{C'}(S')$, and further, given morphisms $F$ and $G$ over $f\colon Y' \to Y$ and $g\colon Y'' \to Y'$, after identifying $(FG)^*\mathcal{O}_C(S)$ with $G^*F^*\mathcal{O}_C(S)$ the isomorphisms $(FG)^*\mathcal{O}_C(S) \to \mathcal{O}_{C''}(S'')$ and $G^*F^*\mathcal{O}_C(S) \to G^*\mathcal{O}_{C'}(S') \to \mathcal{O}_{C''}(S'')$ coincide. Then, by descent theory of flat proper morphisms of schemes with a relatively ample invertible sheaf, see [Vi05, Thm. 4.38], descent data for objects of $\overline{\mathcal{L}}_n$ are effective.

We show that the diagonal $\overline{\mathcal{L}}_n \to \overline{\mathcal{L}}_n \times \overline{\mathcal{L}}_n$ is representable and finite. For a scheme $Y$ and a morphism $Y \to \overline{\mathcal{L}}_n \times \overline{\mathcal{L}}_n$ given by two objects $\mathscr{C}, \mathscr{C}' \in \overline{\mathcal{L}}_n(Y)$, the category $Y \times_{\overline{\mathcal{L}}_n \times \overline{\mathcal{L}}_n} \overline{\mathcal{L}}_n$ fibred over the category of $Y$-schemes is isomorphic to the functor on $Y$-schemes $\mathrm{Isom}(\mathscr{C}, \mathscr{C}')(f\colon Z \to Y) = \mathrm{Mor}_{\overline{\mathcal{L}}_n(Z)}(f^*\mathscr{C}, f^*\mathscr{C}')$. Using the embedding via $\mathcal{O}_C(S)$ into $\mathbb{P}_Y(\pi_*\mathcal{O}_C(S)) \cong \mathbb{P}^n_Y$ described in proposition 3.4 we see that $\mathrm{Isom}(\mathscr{C}, \mathscr{C}')$ is a finite closed subgroup scheme of the open dense torus of $\mathbb{P}^n_Y$. In characteristic 0 it is unramified over $Y$, because then the fibres are reduced. $\square$

The stack $\overline{\mathcal{L}}_n$ is related to the Losev-Manin moduli space $\overline{L}_n$ by a morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$ that arises by considering the $n$ sections of an $n$-pointed chain over $Y$ as a relative effective Cartier divisor of degree $n$ over $Y$.

**Proposition 1.5.** *The morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$ is faithfully flat and finite of degree $n!$. It is ramified exactly in the points of $\overline{L}_n$ corresponding to $n$-pointed chains with some coinciding marked points.*

*Proof.* Note that the morphism is representable, since $\overline{\mathcal{L}}_n$ has representable diagonal. We show that for any morphism $Y \to \overline{\mathcal{L}}_n$, $Y$ a scheme, the morphism of schemes $Y \times_{\overline{\mathcal{L}}_n} \overline{L}_n \to Y$ has the properties in question. The morphism $Y \to \overline{\mathcal{L}}_n$ corresponds to an object $\mathscr{C} = (C \to Y, s_-, s_+, S)$ over $Y$ and the functor $Y \times_{\overline{\mathcal{L}}_n} \overline{L}_n$ maps a scheme $T$ to the set $\{(f\colon T \to Y, (C' \to T, s_-, s_+, s_1, \ldots, s_n), \alpha) \mid \alpha\colon f^*\mathscr{C} \to (C' \to T, s_-, s_+, s_1 + \ldots + s_n)\}$, where $s_1 + \ldots + s_n$ denotes the divisor of degree $n$ associated to the $n$ sections and $\alpha$ is a morphism in $\overline{\mathcal{L}}_n(T)$. We denote by $C_0$ the open subscheme of $C$ obtained by excluding the poles and intersection points of components on the fibres. Then $C_0$ is a quasi-projective curve over $Y$, which is smooth over $Y$ since it is flat with smooth fibres (see [EGA, IV, (17.5.1)]). We may, for any $T$, identify the chains $C'$ over $T$ occurring in the above sets with $C \times_Y T$ via the specified isomorphisms. The additional data given by the subscheme $S \subset C_0$ are equivalent to a section $s\colon Y \to \mathrm{Div}^n_{C_0/Y} = C_0^{(n)}$ of the scheme of relative effective divisors of degree $n$, which coincides with the $n$-fold symmetric product of $C_0$ over $Y$, see [SGA4(3), Exposé XVII, 6.3.9, p. 186]. Likewise, the data given by the sections

$s_1, \dots, s_n$ are equivalent to a section $s' \colon T \to (C_0^n)_T$ of the $n$-fold product such that its composition with $(C_0^n)_T \to (C_0^{(n)})_T$ is the base extension $s_T$ of $s$, or equivalently, to a morphism $s' \colon T \to C_0^n$ whose composition with $C_0^n \to C_0^{(n)}$ coincides with $s \circ f$. Thus the functor $Y \times_{\overline{\mathcal{L}}_n} \overline{L}_n$ is isomorphic to the functor of the scheme $Y \times_{C_0^{(n)}} C_0^n$, and this concludes the proof because the morphism $C_0^n \to C_0^{(n)}$ has the required properties. □

**Remark 1.6.** With proposition 1.5 and some general theory we can derive some properties of the stack $\overline{\mathcal{L}}_n$: by [LMB, Thm. 10.1], making use of the proposition, $\overline{\mathcal{L}}_n$ is an algebraic stack (Artin stack); in characteristic 0, by [LMB, Thm. 8.1] and the fact that it has unramified diagonal, it is a Deligne-Mumford stack. However, the result will follow independently later in section 3 together with a more detailed description of the structure of $\overline{\mathcal{L}}_n$.

On the Losev-Manin moduli space $\overline{L}_n$ we have an operation of the symmetric group $S_n$ permuting the $n$ sections. Any $S_n$-equivariant morphism $\overline{L}_n \to Z$, $Z$ a scheme with trivial $S_n$-action, factors through $\overline{L}_n \to \overline{\mathcal{L}}_n$. This implies that the quotient morphism $\overline{L}_n \to \overline{L}_n/S_n$ factors as

$$\overline{L}_n \;\to\; \overline{\mathcal{L}}_n \;\to\; \overline{L}_n/S_n,$$

and moreover, as $\overline{L}_n \to \overline{\mathcal{L}}_n$ is an epimorphism, $\overline{\mathcal{L}}_n \to \overline{L}_n/S_n$ forms the coarse moduli space of $\overline{\mathcal{L}}_n$.

**Remark 1.7.** There is the quotient stack $[\overline{L}_n/S_n]$, which has the same geometric points as $\overline{\mathcal{L}}_n$. However, the automorphism groups of objects of $[\overline{L}_n/S_n]$, differ from those of $\overline{\mathcal{L}}_n$ which are always abelian.

In the case of the Losev-Manin moduli spaces, the boundary divisors arise as images of closed embeddings $\overline{L}_m \times \overline{L}_n \to \overline{L}_{m+n}$. For the stacks $\overline{\mathcal{L}}_n$ we also have embeddings $\overline{\mathcal{L}}_m \times \overline{\mathcal{L}}_n \to \overline{\mathcal{L}}_{m+n}$, defined as in the Losev-Manin case by concatenation of chains, and the diagrams

$$\begin{array}{ccc} \overline{L}_m \times \overline{L}_n & \longrightarrow & \overline{L}_{m+n} \\ \downarrow & & \downarrow \\ \overline{\mathcal{L}}_m \times \overline{\mathcal{L}}_n & \longrightarrow & \overline{\mathcal{L}}_{m+n} \end{array} \tag{1}$$

are commutative.

The morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$ maps the open dense torus of $X(A_n)$, the moduli space of irreducible $n$-pointed chains, onto the moduli stack of irreducible degree-$n$-pointed chains. This open substack of $\overline{\mathcal{L}}_n$ parametrises subschemes $S$ of degree $n$ in $\mathbb{P}^1 \setminus \{0, \infty\}$ modulo automorphisms of $\mathbb{P}^1$ fixing 0 and $\infty$. An object over an algebraically closed field $K$ can described by a monic polynomial $\prod_{i=1}^n (y - s_i)$ of degree $n$ with $s_1, \dots, s_n \in K^*$ determined up to scaling by a common factor $\lambda \in K^*$ and permutations. We can write this polynomial as

$$y^n \;-\; (s_1 + \ldots + s_n) y^{n-1} \;+\; \quad \ldots\ldots \quad +\; (-)^n s_1 \cdots s_n$$

where the coefficients are the symmetric polynomials in $s_1, \dots, s_n$. Assuming $s_1 \cdots s_n = (-1)^n$, we have a polynomial of the form

$$y^n \;+\; a_1 y^{n-1} \;+\; a_2 y^{n-2} \;+\; \quad \ldots\ldots \quad +\; a_{n-1} y \;+\; 1$$

with coefficients $a_1, \ldots, a_{n-1} \in K$. The isomorphism class of the object determines these coefficients up to the equivalence $(a_1, \ldots, a_{n-1}) \sim (\xi^{n-1}a_1, \ldots, \xi a_{n-1})$, $\xi$ an $n$-th root of unity. The moduli stack of such objects is the quotient stack $[\mathbb{A}^{n-1}/\mu_n]$, where the group scheme $\mu_n$ of $n$-th roots of unity acts with weights $(n-1, \ldots, 1)$.

It contains an $(n-1)$-dimensional algebraic torus $T$ parametrising classes of polynomials with non-zero coefficients. A $K$-valued point of $T$ is given by an $(n-1)$-tuple

$$b_1 = \tfrac{a_2}{a_1^2},\ b_2 = \tfrac{a_1 a_3}{a_2^2},\ \ldots\ ,\ b_k = \tfrac{a_{k-1}a_{k+1}}{a_k^2},\ \ldots\ ,\ b_{n-2} = \tfrac{a_{n-3}a_{n-1}}{a_{n-2}^2},\ b_{n-1} = \tfrac{a_{n-2}}{a_{n-1}^2}$$

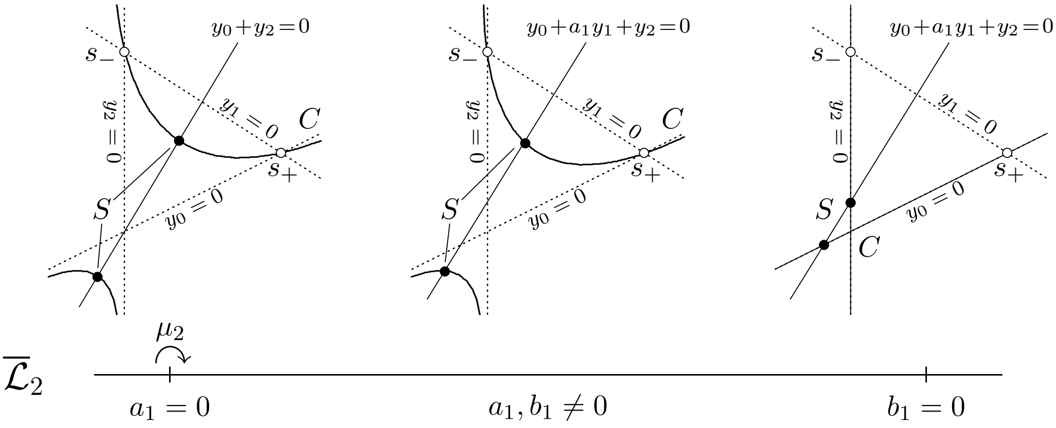
of elements $b_i$ of the field $K$. These expressions in the $a_i$ form a set of generators of $\mu_n$-invariants in the coordinate ring of the torus $(\mathbb{G}_m)^{n-1} \subset \mathbb{A}^{n-1}$. Equivalently, we can express a $K$-valued point of $T$ as a collection $(a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1})$ up to the equivalence

$$(a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}) \sim (\kappa_1 a_1, \ldots, \kappa_{n-1}a_{n-1}, \lambda_i b_1, \ldots, \lambda_{n-1}b_{n-1})$$

for $\kappa_i \in K^*$ and $\lambda_i = \kappa_i^2/(\kappa_{i-1}\kappa_{i+1})$, putting $\kappa_0 = \kappa_n = 1$.

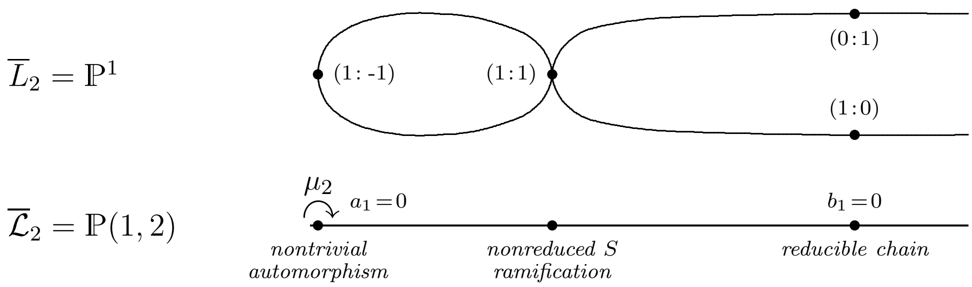
Allowing certain subsets of the coordinates $a_i, b_i$ to become zero, we obtain a toric tame stack which compactifies the moduli stack $[\mathbb{A}^{n-1}/\mu_n]$ of irreducible chains. Its definition and properties are contained in section 2 and we will show in section 3 that it coincides with the moduli stack $\overline{\mathcal{L}}_n$. In particular, degenerating some of the $b_i$ to zero can be interpreted as degenerating a projective line to a reducible chain of projective lines. These additional divisors arise as in diagram (1).

**Example 1.8.** We illustrate some results of this paper in the case $n = 2$ (see also examples 2.2, 3.8, 4.16). There is a natural embedding of degree-2-pointed chains $(C, s_-, s_+, S)$ into $\mathbb{P}^2$ determined by the line bundle $\mathcal{O}_C(S)$ (for arbitrary $n$ see section 3): in $\mathbb{P}^2 = \mathbb{P}(H^0(C, \mathcal{O}_C(S)))$ we can choose homogeneous coordinates $y_0, y_1, y_2$ such that $C$ is given by an equation $y_0 y_2 = b_1 y_1^2$, the subscheme $S \subset C$ by an additional equation $y_0 + a_1 y_1 + y_2 = 0$, and the two sections $s_-, s_+$ are $(1:0:0), (0:0:1)$. Over an algebraically closed field $K$, data $(a_1, b_1) \in K^2 \setminus \{(0,0)\}$ up to the equivalence $(a_1, b_1) \sim (\kappa_1 a_1, \lambda_1 b_1)$ for $\kappa_1^2 = \lambda_1 \in K^*$ correspond to isomorphism classes of degree-2-pointed chains over $K$.



The moduli stack $\overline{\mathcal{L}}_2$ is isomorphic to the quotient stack $[(\mathbb{A}^2 \setminus \{(0,0)\})/\mathbb{G}_m]$ for the operation with weights $(1,2)$, i.e. the weighted projective line $\mathbb{P}(1,2)$ (which coincides with the toric orbifold $\mathcal{Y}(A_{n-1})$ for $n=2$ defined in section 2). The open substack parametrising irreducible curves, the locus where $b_1 \neq 0$, is the quotient stack $[\mathbb{A}^1/\mu_2]$ with coordinate $a_1$ on $\mathbb{A}^1$. The open substack parametrising objects without isomorphisms, the locus where $a_1 \neq 0$, is isomorphic to $\mathbb{A}^1$ with coordinate $b_1$.

The morphism $\overline{L}_2 \to \overline{\mathcal{L}}_2$ is faithfully flat and finite of degree 2. We introduce homogeneous coordinates $z_-, z_+$ of $\overline{L}_2 = \mathbb{P}^1$ that measure the position of one of the marked points of a 2-pointed chain with respect to the other marked point at $(1:1)$ of its component isomorphic to $\mathbb{P}^1$, such that the two points $(0:1), (1:0)$ correspond to reducible chains (cf. [BB11a]). Then the point $(1:\text{-}1)$ corresponds to a 2-pointed curve $\mathbb{P}^1$ with marked points $(1:1)$, $(1:\text{-}1)$ giving rise to a degree-2-pointed curve with automorphism group $\mu_2$. The point $(1:1)$ corresponds to the point of $\overline{\mathcal{L}}_2$ with nonreduced $S$, the morphism is ramified here and étale elsewhere.




We will see in section 4 that the morphism $\overline{L}_2 \to \overline{\mathcal{L}}_2$ is given as

$$(z_- + z_+ : z_- z_+) : \ \mathbb{P}^1 \ \to \ \mathbb{P}(1,2).$$

## 2. The toric orbifolds $\mathcal{Y}(A_n)$

In this section we will consider a family of toric orbifolds associated to the Cartan matrices of root systems of type $A$, but also comment on some generalities on toric stacks.

We use the definitions and notations of [BCS05]. A stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ defining a toric orbifold has the property that the abelian group $N$ is free; it consists of the data of a simplicial fan $\Sigma$ in the lattice $N$ and elements $n_\varrho \in \varrho \cap N$ for the one-dimensional cones $\varrho \in \Sigma(1)$. Here we assume them to span the ambient space $N_\mathbb{Q}$. The homomorphism $\beta \colon \mathbb{Z}^{\Sigma(1)} \to N$ maps the elements of the standard basis to the elements $n_\varrho$. Dually we have the exact sequence

$$0 \longrightarrow M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z}) \xrightarrow{\beta^*} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\beta^\vee} \mathrm{DG}(\beta) \longrightarrow 0$$

giving rise, as sequence of character groups, to the exact sequence of diagonalisable group schemes

$$1 \longrightarrow G \longrightarrow T_{\Sigma(1)} \longrightarrow T_M \longrightarrow 1.$$

The toric orbifold $\mathcal{X}_{\mathbf{\Sigma}}$ is defined as the quotient stack $[U/G]$ with $U \subseteq \mathbb{A}^{\Sigma(1)}$ the open subset defined by the information which of the one-dimensional cones form higher dimensional cones of $\Sigma$. The constructions make sense over the integers, however, working with $G$-torsors, in general one may have to choose an appropriate Grothendieck topology on the base category possibly finer than the étale topology (see also remark 2.8). The resulting algebraic stacks $\mathcal{X}_{\mathbf{\Sigma}}$ are tame stacks in the sense of [AOV08].

**Definition 2.1.** We define the toric orbifold $\mathcal{Y}(A_n)$ associated to the Cartan matrix of the root system $A_n$ in terms of the stacky fan $\mathbf{\Upsilon}(A_n) = (N, \Upsilon(A_n), \beta)$: let $N = \mathbb{Z}^n$ and let the linear map $\beta\colon \mathbb{Z}^{2n} \to N$ be given by the $n \times 2n$ matrix

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & \cdots & 0 & 1 & 0 & \cdots \\ 0 & 1 & -2 & \cdots & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. the matrix consisting of two blocks ($-C(A_n)\ I_n$), where $C(A_n)$ is the Cartan matrix of the root system $A_n$ and $I_n$ the $n \times n$ identity matrix. The fan $\Upsilon(A_n)$ consists of the $2n$ one-dimensional cones $\varrho_1, \dots, \varrho_n, \tau_1, \dots, \tau_n$ generated by the columns of the above matrix. A subset of one-dimensional cones generates a higher dimensional cone if it does not contain one of the sets $\{\varrho_1, \tau_1\}, \dots, \{\varrho_n, \tau_n\}$. This defines a fan containing $2^n$ $n$-dimensional cones $\sigma_I$ generated by sets $\{\varrho_i : i \notin I\} \cup \{\tau_i : i \in I\}$ for subsets $I \subseteq \{1, \dots, n\}$.

For the stacky fan $\mathbf{\Upsilon}(A_n)$ the map $\beta\colon \mathbb{Z}^{2n} \to N$ gives rise to the exact sequence of lattices

$$0 \longrightarrow M \cong \mathbb{Z}^n \xrightarrow{\binom{-C}{I_n}} \mathbb{Z}^{2n} \xrightarrow{(I_n\ C)} \mathrm{DG}(\beta) \cong \mathbb{Z}^n \longrightarrow 0$$

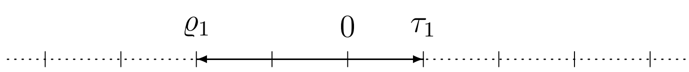
where $C = C(A_n)^\top = C(A_n)$ is (the transpose of) the Cartan matrix, and the exact sequence of tori

$$1 \longrightarrow G \cong (\mathbb{G}_m)^n \longrightarrow (\mathbb{G}_m)^{2n} \longrightarrow T_N \cong (\mathbb{G}_m)^n \longrightarrow 1$$

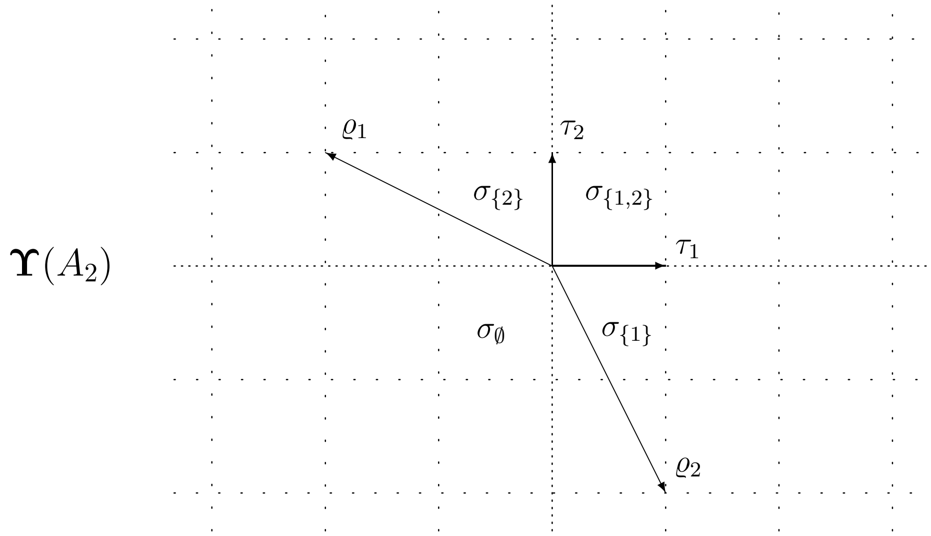
where $G \cong (\mathbb{G}_m)^n \longrightarrow (\mathbb{G}_m)^{2n}$, $(\kappa_1, \dots, \kappa_n) \mapsto (\kappa_1, \dots, \kappa_n, \lambda_1, \dots, \lambda_n)$ with $\lambda_i = \kappa_i^2/(\kappa_{i-1}\kappa_{i+1})$ setting $\kappa_0 = \kappa_{n+1} = 1$ (cf. last section). Note that the toric orbifold $\mathcal{Y}(A_n)$ arises as quotient $[U/G]$ by a torus $G$.

**Example 2.2.** The toric orbifold $\mathcal{Y}(A_1)$ is isomorphic to the weighted projective line $\mathbb{P}(1, 2)$: we have the matrix $(\,-2\ \ 1\,)$ and the stacky fan looks as follows:

$\mathbf{\Upsilon}(A_1)$



**Example 2.3.** The toric orbifold $\mathcal{Y}(A_2)$ arises from the matrix $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}$. We have the stacky fan




The description of the functor of a smooth toric variety given by Cox [Co95b] in terms of collections of line bundles with sections determined by the combinatorial data has been extended to toric Deligne-Mumford stacks by Iwanari [Iw07] and Perroni [Pe08]. For the stacky fan $\mathbf{\Upsilon}(A_n)$ we have:

**Definition 2.4.** An $\mathbf{\Upsilon}(A_n)$-collection on a scheme $Y$ is a collection

$$\mathscr{L} = ((\mathscr{L}_{\varrho_i}, a_i)_{i=1,\ldots,n}, (\mathscr{L}_{\tau_i}, b_i)_{i=1,\ldots,n}, (c_i)_{i=1,\ldots,n})$$

where $(\mathscr{L}_{\varrho_i}, a_i)$ and $(\mathscr{L}_{\tau_i}, b_i)$ are line bundles with a section and

$$c_1\colon \mathscr{L}_{\tau_1}\otimes\mathscr{L}_{\varrho_1}^{\otimes\text{-}2}\otimes\mathscr{L}_{\varrho_2} \to \mathcal{O}_Y,\ c_2\colon \mathscr{L}_{\tau_2}\otimes\mathscr{L}_{\varrho_1}\otimes\mathscr{L}_{\varrho_2}^{\otimes\text{-}2}\otimes\mathscr{L}_{\varrho_3} \to \mathcal{O}_Y, \quad \ldots\ldots\ldots\ldots$$
$$\ldots\ldots \quad c_{n-1}\colon \mathscr{L}_{\tau_{n-1}}\otimes\mathscr{L}_{\varrho_{n-2}}\otimes\mathscr{L}_{\varrho_{n-1}}^{\otimes\text{-}2}\otimes\mathscr{L}_{\varrho_n} \to \mathcal{O}_Y,\ c_n\colon \mathscr{L}_{\tau_n}\otimes\mathscr{L}_{\varrho_{n-1}}\otimes\mathscr{L}_{\varrho_n}^{\otimes\text{-}2} \to \mathcal{O}_Y$$

are isomorphisms. These data are subject to the nondegeneracy condition that for every point $y \in Y$ and $i = 1, \ldots, n$ not both $a_i(y) = 0$ and $b_i(y) = 0$.

A morphism $\mathscr{L}' \to \mathscr{L}$ between two $\mathbf{\Upsilon}(A_n)$-collections $\mathscr{L} = ((\mathscr{L}_{\varrho_i}, a_i)_i, (\mathscr{L}_{\tau_i}, b_i)_i, (c_i)_i)$ on $Y$ and $\mathscr{L}' = ((\mathscr{L}'_{\varrho_i}, a'_i)_i, (\mathscr{L}'_{\tau_i}, b'_i)_i, (c'_i)_i)$ on $Y'$ over a morphism of schemes $f\colon Y' \to Y$ is a collection $((r_i)_{i=1,\ldots,n}, (t_i)_{i=1,\ldots,n})$ consisting of isomorphisms of line bundles $r_i\colon f^*\mathscr{L}_{\varrho_i} \to \mathscr{L}'_{\varrho_i}$, $t_i\colon f^*\mathscr{L}_{\tau_i} \to \mathscr{L}'_{\tau_i}$ such that $r_i(f^*a_i) = a'_i$, $t_i(f^*b_i) = b'_i$ and the diagrams

$$\begin{array}{ccc} f^*\mathscr{L}_{\tau_i}\otimes f^*\mathscr{L}_{\varrho_{i-1}}\otimes f^*\mathscr{L}_{\varrho_i}^{\otimes\text{-}2}\otimes f^*\mathscr{L}_{\varrho_{i+1}} & \xrightarrow{f^*c_i} & f^*\mathcal{O}_Y \\ \downarrow & & \downarrow \\ \mathscr{L}'_{\tau_i}\otimes\mathscr{L}'_{\varrho_{i-1}}\otimes{\mathscr{L}'_{\varrho_i}}^{\otimes\text{-}2}\otimes\mathscr{L}'_{\varrho_{i+1}} & \xrightarrow{c'_i} & \mathcal{O}_{Y'} \end{array} \tag{2}$$

($i = 1, \ldots, n$; for $i = 1, n$ omit the factors indexed by $\varrho_0, \varrho_{n+1}$) commute.

We denote the fibred category of $\mathbf{\Upsilon}(A_n)$-collections over the category of schemes by $\mathcal{C}_{\mathbf{\Upsilon}(A_n)}$. It comes with the cleavage given by pull-back of line bundles: for $f: Y' \to Y$ we have an arrow $f^*\mathscr{L} \to \mathscr{L}$ in $\mathcal{C}_{\mathbf{\Upsilon}(A_n)}$. The definition describes a morphism $\mathscr{L}' \to \mathscr{L}$ in $\mathcal{C}_{\mathbf{\Upsilon}(A_n)}$ as composition of a morphism $\mathscr{L}' \to f^*\mathscr{L}$ over $id_Y$ with $f^*\mathscr{L} \to \mathscr{L}$ over $f: Y' \to Y$.

**Remark 2.5.** For an affine scheme $Y$ a morphism of $\mathbf{\Upsilon}(A_n)$-collections $\mathscr{L}' \to \mathscr{L}$ over $id_Y$, after fixing isomorphisms of the line bundles with the structure sheaf, corresponds to a collection $\kappa_1, \dots, \kappa_n, \lambda_1, \dots, \lambda_n \in \mathcal{O}_Y^*(Y)$ such that the isomorphisms $\mathcal{O}_Y \cong \mathscr{L}_{\varrho_i} \to \mathscr{L}'_{\varrho_i} \cong \mathcal{O}_Y$ and $\mathcal{O}_Y \cong \mathscr{L}_{\tau_i} \to \mathscr{L}'_{\tau_i} \cong \mathcal{O}_Y$ are given by multiplication by $\kappa_i$ and $\lambda_i$. The condition expressed in diagram (2) translates into the equations $\lambda_i = \kappa_i^2/(\kappa_{i-1}\kappa_{i+1})$, putting $\kappa_0 = \kappa_{n+1} = 1$.

The category of $\mathbf{\Sigma}$-collections $\mathcal{C}_{\mathbf{\Sigma}}$ for a stacky fan $\mathbf{\Sigma}$ is a category fibred in groupoids (CFG) over the base category of schemes. By descent theory for quasi-coherent sheaves the CFG $\mathcal{C}_{\mathbf{\Sigma}}$ forms a stack in the fpqc topology, see [Vi05, Thm. 4.23]. By Iwanari [Iw07, Thm. 1.4] (for toric orbifolds) and Perroni [Pe08, Thm. 2.6] (for toric Deligne-Mumford stacks) over fields of characteristic 0, working with the étale topology, there is an isomorphism of stacks $\mathcal{X}_{\mathbf{\Sigma}} \cong \mathcal{C}_{\mathbf{\Sigma}}$. Also over more general base schemes we have an isomorphism $\mathcal{X}_{\mathbf{\Sigma}} \cong \mathcal{C}_{\mathbf{\Sigma}}$; we make some comments on this issue.

**Construction 2.6.** Explicitely, one can construct an isomorphism $\mathcal{X}_{\mathbf{\Sigma}} \cong \mathcal{C}_{\mathbf{\Sigma}}$ as follows, here for simplicity we stick to the orbifold case and assume that the one-dimensional cones generate the ambient space $N_{\mathbb{Q}}$.

Note that we have a natural $G$-equivariant $\mathbf{\Sigma}$-collection $((\mathcal{O}_U \otimes V_\varrho, x_\varrho)_\varrho, (id)_m)$ on $U$, where $V_\varrho$ is the one-dimensional representation such that the coordinate $x_\varrho$ of $U \subset \mathbb{A}^{\Sigma(1)}$ is an invariant section of $\mathcal{O}_U \otimes V_\varrho$ (in the case of smooth toric varieties as considered in [Co95a] this $G$-equivariant collection descents to the universal collection on the toric variety). Since both fibred categories are Zariski-stacks we can restrict to the base category of affine schemes.

Starting with an object of $\mathcal{X}_{\mathbf{\Sigma}}$ over $Y$, that is a $G$-torsor $p\colon E \to Y$ together with a $G$-equivariant morphism $t\colon E \to U$, the pull-back $t^*((\mathcal{O}_U \otimes V_\varrho, x_\varrho)_\varrho, (id)_m)$ is a $G$-equivariant $\mathbf{\Sigma}$-collection on $E$ and gives rise to the $\mathbf{\Sigma}$-collection $p_*^G t^*((\mathcal{O}_U \otimes V_\varrho, x_\varrho)_\varrho, (id)_m)$ on $Y$ (the functor $p_*^G$ takes the $G$-invariant part of the push-forward).

On the other hand, for a given $\mathbf{\Sigma}$-collection $((\mathscr{L}_\varrho, u_\varrho)_\varrho, (c_m)_m)$ on an affine scheme $Y$ we construct a $G$-torsor with a $G$-equivariant morphism to $U$. Let $\underline{E}$ be the contravariant functor on the category of $Y$-schemes

$$\underline{E}:\ (q\colon Y' \to Y) \ \mapsto \ \left\{ \begin{array}{l} \mathbf{\Sigma}\textit{-collections } ((\mathcal{O}_{Y'} \otimes V_\varrho, u'_\varrho), (id)_m) \textit{ on } Y' \\ \textit{with an isomorphism of } \mathbf{\Sigma}\textit{-collections} \\ q^*((\mathscr{L}_\varrho, u_\varrho)_\varrho, (c_m)_m) \cong ((\mathcal{O}_{Y'} \otimes V_\varrho, u'_\varrho)_\varrho, (id)_m) \end{array} \right\}$$

where $V_\varrho$, the one-dimensional representation as above, is used to define an operation of $G$ on this functor. Then one can show that the functor $\underline{E}$ with this $G$-action is represented by a $G$-torsor $p\colon E \to Y$ together with a universal isomorphism $p^*((\mathscr{L}_\varrho, u_\varrho)_\varrho, (c_m)_m) \cong ((\mathcal{O}_E \otimes V_\varrho, u_\varrho^E)_\varrho, (id)_m)$ of $G$-equivariant $\mathbf{\Sigma}$-collections, provided that the original $\mathbf{\Sigma}$-collection is locally trivial in the sense that there is a

covering $f\colon Y' \to Y$ such that $f^*((\mathscr{L}_\varrho, u_\varrho)_\varrho, (c_m)_m)$ is isomorphic to a collection of the form $((\mathcal{O}_{Y'}, u'_\varrho)_\varrho, (id)_m)$; collections of this form correspond to trivial $G$-torsors. We will assume that the topology on the base category is such that any $\mathbf{\Sigma}$-collection has this property, see also the following remarks. The sections $(u^E_\varrho)_\varrho$ of the universal $\mathbf{\Sigma}$-collection on $E$ then define a $G$-equivariant morphism $E \to U \subset \mathbb{A}^{\Sigma(1)}$.

Making use of the fact that for a $G$-torsor $p\colon E \to Y$ we have the equivalence $\mathrm{QCoh}(Y) \leftrightarrow \mathrm{QCoh}^G(E)$ given by the functors $p^*$ and $p^G_*$, one can show that these constructions define functors $\mathcal{X}_{\mathbf{\Sigma}} \leftrightarrow \mathcal{C}_{\mathbf{\Sigma}}$ whose compositions are isomorphic to the identity functors.

**Remark 2.7.** We can interpret the construction of the $G$-torsor as the coboundary homomorphism $d$ in the exact sequence (see [Gi, Ch. III, §3])

$$0 \longrightarrow H^0(Y, G) \longrightarrow H^0(Y, T_{\Sigma(1)}) \longrightarrow H^0(Y, T_M) \xrightarrow{d} H^1(Y, G)$$

where elements of $H^1(Y, G)$ are isomorphism classes of $G$-torsors over $Y$: given a $\mathbf{\Sigma}$-collection $((\mathcal{O}_Y, u_\varrho)_\varrho, (c_m)_m)$ on $Y$, the automorphisms $(c_m)_m$ of the structure sheaf can be interpreted as a morphism $Y \to T_M$ or section of $T_M \times Y \to Y$, and fitting into the cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & T_{\Sigma(1)} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & T_M \end{array}$$

we obtain a $G$-torsor $E \to Y$ which is trivial if and only if $(c_m)_m \in H^0(Y, T_M)$ comes from an element of $H^0(Y, T_{\Sigma(1)})$.

**Remark 2.8.** Working with $G$-torsors, we usually assume that the Grothendieck topology on the base category is fine enough in the sense that we have the same $G$-torsors as we have with respect to the canonical topology. We have seen that this assumption was necessary to derive the isomorphism $\mathcal{X}_{\mathbf{\Sigma}} \cong \mathcal{C}_{\mathbf{\Sigma}}$: whereas the notion of $G$-torsor depends on the topology, this is not the case for the notion of $\mathbf{\Sigma}$-collections. For $\mathbf{\Sigma}$-collections we have the corresponding assumption that $\mathbf{\Sigma}$-collections are locally trivial with respect to the topology (in the sense of construction 2.6). In characteristic 0 this is always true for the étale topology. In general we may have to take a finer topology, for example the fppf topology.

In the case of the stacky fan $\mathbf{\Upsilon}(A_n)$ the lattice $M$ is a direct summand of $\mathbb{Z}^{\Sigma(1)}$ and the group scheme $G$ a torus, so the following result also holds in weaker topologies like étale or Zariski.

**Corollary 2.9.** *There is an isomorphism of stacks* $\mathcal{Y}(A_n) \cong \mathcal{C}_{\mathbf{\Upsilon}(A_n)}$.

In particular, a $K$-valued point of $\mathcal{Y}(A_n)$ corresponds to $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in K^{2n}$ such that for any $i$ not both $a_i = 0$ and $b_i = 0$, up to the equivalence relation given by multiplication by a collection $(\kappa_1, \ldots, \kappa_n, \lambda_1, \ldots, \lambda_n) \in (K^*)^{2n}$ as in remark 2.5.

## 3. $\mathcal{Y}(A_{n-1})$ AS MODULI STACK OF DEGREE-$n$-POINTED CHAINS

In this section we will prove the following theorem.

**Theorem 3.1.** *There is an isomorphism of stacks $\overline{\mathcal{L}}_n \cong \mathcal{Y}(A_{n-1})$.*

Because the fibred categories under consideration are stacks we can restrict to the base category of affine schemes. We will relate families of pointed chains over affine schemes to $\boldsymbol{\Upsilon}(A_{n-1})$-collections and prove an equivalence of fibred categories $\overline{\mathcal{L}}_n \cong \mathcal{C}_{\boldsymbol{\Upsilon}(A_{n-1})}$.

Let $(C, s_-, s_+, S)$ be a degree-$n$-pointed chain of projective lines over a field $K$. We look at the closed embedding $C \to \mathbb{P}_K(H^0(C, \mathcal{O}_C(S))) \cong \mathbb{P}^n_K$ determined by $\mathcal{O}_C(S)$.

First assume that $C$ is irreducible, that is $C \cong \mathbb{P}^1_K$. The vector space $H^0(C, \mathcal{O}_C(S))$ is $(n+1)$-dimensional and we have a basis $y_0, \dots, y_n$ such that the divisor of $y_i$ satisfies $\operatorname{div}(y_i) = is_- + (n-i)s_+$. The ideal sheaf $\mathscr{I} = \mathcal{O}_C(-S) \to \mathcal{O}_C$ defining $S$ is a line bundle. Tensored with $\mathcal{O}_C(S)$ we have an inclusion $\mathcal{O}_C \to \mathcal{O}_C(S)$ with cokernel $\mathcal{O}_S$, and the image of the 1-section of $\mathcal{O}_C$ is a global section $\sum_{i=0}^n a_i y_i \in H^0(Y, \mathcal{O}_C(S))$. The subscheme $S \subset C$ is given by the equation $\sum_{i=0}^n a_i y_i = 0$, where $a_0, a_n \neq 0$ as $S$ does not meet $s_-, s_+$. We can choose the basis $y_0, \dots, y_n$ such that $a_0 = a_n = 1$.

The embedding defined by $\mathcal{O}_C(S)$, the $n$-fold Veronese embedding or $n$-uple embedding, gives an isomorphism of $C$ onto the subscheme in $\mathbb{P}^n_K$ determined by the equations

$$y_i y_{j+1} = b_{i+1} \cdots b_j y_{i+1} y_j$$

for $0 \le i < j < n$ and certain numbers $b_1, \dots, b_{n-1} \in K^*$. These equations express the condition that the rank of the matrix

$$\begin{pmatrix} y_0 & b_1 y_1 & \dots & b_1 \cdots b_{n-1} y_{n-1} \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$$

is less than 2. The subscheme $S$ in the embedded curve is given by the additional linear equation

$$y_n \;+\; a_{n-1} y_{n-1} \;+\; \dots \;+\; a_1 y_1 \;+\; y_0 \;=\; 0.$$

Similarly, we have a natural embedding of reducible degree-$n$-pointed chains of projective lines into $\mathbb{P}^n_K$.

**Proposition 3.2.** *Let $(C, s_-, s_+, S)$ be a degree-$n$-pointed chain of projective lines over a field $K$. It decomposes into irreducible components $C_1, \dots, C_m \cong \mathbb{P}^1_K$ with poles $(p_1^-, p_1^+), \dots, (p_m^-, p_m^+)$ such that $s_- = p_1^-$, $s_+ = p_m^+$ and $C_i$ intersects $C_{i+1}$ in $p_i^+ = p_{i+1}^-$. Let $n_1, \dots, n_m$ be the degrees of $S$ on the components $C_1, \dots, C_m$ and $N_k = \sum_{i=1}^k n_i$. Then there is a basis $y_0, \dots, y_n$ of $H^0(C, \mathcal{O}_C(S))$ characterised up to nonzero scalars by the following conditions: $y_i$ is nonzero only on the components $C_k$ satisfying $N_{k-1} \le i \le N_k$ and in this case $\operatorname{div}(y_i)|_{C_k} = (i - N_{k-1})p_k^- + (N_k - i)p_k^+$. We scale $y_0, y_n$ such that the image of the 1-section under the inclusion $\mathcal{O}_C \to \mathcal{O}_C(S)$ is $\sum_{i=0}^n a_i y_i \in H^0(Y, \mathcal{O}_C(S))$ with $a_0 = a_n = 1$, that is, $S$ is given by an equation*

$$y_n \;+\; a_{n-1} y_{n-1} \;+\; \dots \;+\; a_1 y_1 \;+\; y_0 \;=\; 0 \tag{3}$$

*for some $a_1, \dots, a_{n-1} \in K$. These sections $y_0, \dots, y_n$ satisfy the equations*

$$y_i y_{j+1} = b_{i+1} \cdots b_j y_{i+1} y_j \tag{4}$$

*for $0 \le i < j < n$ and certain numbers $b_1, \dots, b_{n-1} \in K$ such that $b_j = 0$ exactly if $j \in \{N_1, \dots, N_{m-1}\}$.*

*The curve $C$ embeds into $\mathbb{P}_K(H^0(C, \mathcal{O}_C(S)))$, the image being the subscheme defined by the equations (4). The subscheme $S$ of the embedded curve is given by the additional equation (3). The sections $s_-, s_+$ are $(1{:}0{:}\dots{:}0)$, $(0{:}\dots{:}0{:}1)$.*

*The numbers $a_1, \dots, a_{n-1}$ and $b_1, \dots, b_{n-1}$ in (3) and (4) have the property that not both $a_j = 0$ and $b_j = 0$.*

We will not work out the proof in detail, but add some remarks.

**Remark 3.3.** The component $C_k$ is embedded into the projective subspace of $\mathbb{P}_K(H^0(C, \mathcal{O}_C(S)))$ spanned by the coordinates $y_{N_{k-1}}, \dots, y_{N_k}$ via a Veronese embedding, the image is given by the equations corresponding to the condition that the rank of the matrix

$$\begin{pmatrix} y_{N_{k-1}} & b_{N_{k-1}+1} y_{N_{k-1}+1} & \dots & b_{N_{k-1}+1} \cdots b_{N_k - 1} y_{N_k - 1} \\ y_{N_{k-1}+1} & y_{N_{k-1}+2} & \dots & y_{N_k} \end{pmatrix}$$

is less than 2. The equation (3) reduces on $C_k$ to $a_{N_k} y_{N_k} + \dots + a_{N_{k-1}} y_{N_{k-1}} = 0$ which defines a finite subscheme $S_k$ of degree $n_k$ in $C_k \subseteq \mathbb{P}^{n_k}_K$. A subscheme $S_k$ of this form does not meet the poles of $C_k$ provided that $a_{N_k}, a_{N_{k-1}} \neq 0$.

We generalise this to degree-$n$-pointed chains over affine schemes.

**Proposition 3.4.** *Let $(\pi\colon C \to Y, s_-, s_+, S)$ be a degree-$n$-pointed chain of projective lines over an affine scheme $Y$. Then there is a decomposition $\pi_* \mathcal{O}_C(S) \cong \bigoplus_{i=0}^n \mathcal{O}_Y y_i$ characterised on the fibres by the properties of proposition 3.2. The generators $y_0, \dots, y_n \in H^0(Y, \pi_* \mathcal{O}_C(S))$ of the individual summands, after possibly rescaling by a global section of $\mathcal{O}_Y^*$, satisfy:*

(i) *The image of the 1-section under the inclusion $\mathcal{O}_C \to \mathcal{O}_C(S)$ is of the form*

$$y_n \;+\; a_{n-1} y_{n-1} \;+\; \dots \;+\; a_1 y_1 \;+\; y_0.$$

(ii) *The kernel of the homomorphism of algebras $\operatorname{Sym} \pi_*(\mathcal{O}_C(S)) \to \bigoplus_{k=0}^\infty \pi_* \mathcal{O}_C(kS)$ is generated by the equations*

$$y_i y_{j+1} = b_{i+1} \cdots b_j y_{i+1} y_j$$

*for $0 \le i < j < n$ and some $b_1, \dots, b_{n-1} \in \mathcal{O}_Y(Y)$.*

*The line bundle $\mathcal{O}_C(S)$ determines a closed embedding $C \to C' \subset \mathbb{P}_Y(\pi_*(\mathcal{O}_C(S)))$ $\cong \mathbb{P}^n_Y$ over $Y$. Its image $C'$ in the coordinates $y_0, \dots, y_n$ is defined by the equations in (ii), the image of $S$ in $C'$ by the additional equation in (i), and the sections $s_-, s_+$ are $(1{:}0{:}\dots{:}0)$, $(0{:}\dots{:}0{:}1)$.*

*Proof.* Given a point $y \in Y$, we construct a decomposition of $\pi_*\mathcal{O}_C(S)$ into a direct sum of structure sheaves over an open subscheme containing $y$. The decomposition of $C_y$ into irreducible components $C_y = C_1 \cup \ldots \cup C_m$ determines over an open subscheme $Y' \subseteq Y$ containing $y$ a decomposition of $S$ into divisors $S_1, \ldots, S_m$ which are disjoint and such that $S_k$ only meets one component on each fibre and the component $C_k$ over $y$. Each $S_k$ determines a morphism onto a $\mathbb{P}^1$-bundle $\mathbb{P}^1_{Y'}$, which on the fibres is an isomorphism on the component containing $S_k$ and contracts the other components (similar as the contraction morphisms in [Kn83], cf. also [BB11a, 3.3]). We have global sections $y_0^{(k)}, \ldots, y_n^{(k)}$ of $\mathcal{O}_{\mathbb{P}^1_{Y'}}(S_k)$ with the property that $y_{N_{k-1}}^{(k)}, \ldots, y_{N_k}^{(k)}$ satisfy $\operatorname{div}(y_i^{(k)}) = (i - N_{k-1})s_- + (N_k - i)s_+$ (using the notation $S_k, s_-, s_+$ also for their images in $\mathbb{P}^1_{Y'}$). Let the other $y_i^{(k)}$ be constant nonzero. Using the pull-backs of these to $C|_{Y'}$ denoted by the same symbols, we define $y_i = y_i^{(1)} \cdots y_i^{(m)} \in H^0(C|_{Y'}, \mathcal{O}_C(S)) = H^0(C|_{Y'}, \mathcal{O}_C(S_1) \otimes \ldots \otimes \mathcal{O}_C(S_m))$. These sections $y_0, \ldots, y_n$ define a decomposition with the required properties over $Y'$. The decompositions $\pi_*\mathcal{O}_C(S)|_{Y'} \cong \bigoplus_{i=0}^n \mathcal{O}_{Y'} y_i$ over the open subschemes $Y' \subseteq Y$ determine a decomposition of $\pi_*\mathcal{O}_C(S)$.

The image of the 1-section under the inclusion $\mathcal{O}_C \to \mathcal{O}_C(S)$ gives a global section $\sum_{i=0}^n a_i y_i \in H^0(Y, \pi_*\mathcal{O}_C(S))$ with $a_i \in \mathcal{O}_Y(Y)$. Since $a_0, a_n \in \mathcal{O}_Y^*(Y)$ we can assume that $a_0, a_n = 1$.

Using what is known about the fibres and results from [EGA, III] (cf. also [Kn83]), we derive that $\pi_*(\mathcal{O}_C(kS))$ for $k > 0$ is locally free of rank $kn + 1$, further that the homomorphism $\pi^*\pi_*\mathcal{O}_C(S) \to \mathcal{O}_C(S)$ is surjective and defines a closed embedding $C \to \mathbb{P}_Y(\pi_*\mathcal{O}_C(S)) \cong \mathbb{P}^n_Y$.

The embedding $C \to \mathbb{P}^n_Y$ corresponds to the surjection of graded algebras $\operatorname{Sym} \pi_*(\mathcal{O}_C(S)) \to \bigoplus_{k=0}^\infty \pi_*\mathcal{O}_C(kS)$. Its kernel $\mathscr{I}$ is the graded ideal that defines the embedded curve $C' \subset \mathbb{P}^n_Y$. Each part $\mathscr{I}_k$ of $\mathscr{I}$ is locally free, being the kernel of a surjective homomorphism of locally free sheaves. The graded ideal $\mathscr{I}$ is generated in degree 2 since this is the case on the fibres $\mathscr{I} \otimes \kappa(y)$ for each point $y \in Y$. The part $\mathscr{I}_2$ of degree 2 is a vector bundle of rank $\frac{1}{2}n(n-1)$. For $i < j$ the subsheaves $\langle y_{i+1}y_j, y_i y_{j+1}\rangle$ and $\langle y_{i+1}y_j\rangle$ of $\pi_*\mathcal{O}_C(S)$, i.e. the subsheaves generated by the respective sections, coincide as this is true on the fibres. Considering the case $j = i+1$, the kernel of the surjective homomorphism $\operatorname{Sym}^2 \pi_*(\mathcal{O}_C(S)) \supset \mathcal{O}_Y y_{i+1}^2 \oplus \mathcal{O}_Y y_i y_{i+2} \to \left\langle y_{i+1}^2, y_i y_{i+2}\right\rangle \subset \pi_*\mathcal{O}_C(2S)$ is a line bundle, so isomorphic to $\mathcal{O}_Y$ and generated by a global section $y_i y_{i+2} - b_{i+1} y_{i+1}^2$ for some $b_{i+1} \in \mathcal{O}_Y(Y)$. For general $i < j$ we have as kernel $y_i y_{j+1} - b_{i+1,j} y_{i+1} y_j$ for some $b_{i+1,j} \in \mathcal{O}_Y(Y)$ and from the equation $(y_{i+2} \cdots y_j) b_{i+1,j} y_{i+1} y_j = (y_{i+2} \cdots y_j) y_i y_{j+1} = (b_{i+1} \cdots b_j)(y_{i+2} \cdots y_j) y_{i+1} y_j$ in $\pi_*\mathcal{O}_C(S)$ we conclude that $b_{i+1,j} = b_{i+1} \cdots b_j$. □

We define a morphism of fibred categories $\overline{\mathcal{L}}_n \to \mathcal{C}_{\boldsymbol{\Upsilon}(A_{n-1})}$.

**Construction 3.5.** Let $\mathscr{C} = (C \to Y, s_-, s_+, S)$ be a degree-$n$-pointed chain of projective lines over an affine scheme $Y$. We construct a decomposition $\mathcal{O}_Y^{\oplus n+1} \cong \pi_*\mathcal{O}_C(S)$ and a basis $y_0, \ldots, y_n$ as in proposition 3.4, and obtain

functions $a_1,\ldots,a_{n-1},b_1,\ldots,b_{n-1}\in\mathcal{O}_Y(Y)$. We define $(\mathscr{L}_{\varrho_i},a_i):=(\mathcal{O}_Y,a_i)$, $(\mathscr{L}_{\tau_i},b_i):=(\mathcal{O}_Y,b_i)$ and have the isomorphisms $c_i\colon\mathscr{L}_{\tau_i}\otimes\mathscr{L}_{\varrho_{i-1}}\otimes\mathscr{L}_{\varrho_i}^{\otimes\text{-}2}\otimes\mathscr{L}_{\varrho_{i+1}}\to\mathcal{O}_Y$ (again omit $\mathscr{L}_{\varrho_0}$, $\mathscr{L}_{\varrho_n}$) given by the identities on $\mathcal{O}_Y$. These data form a $\boldsymbol{\Upsilon}(A_{n-1})$-collection $\mathscr{L}$, the nondegeneracy condition that not both $a_i=0$ and $b_i=0$ in each point is satisfied by construction and proposition 3.2. Different choices of bases $y_0,\ldots,y_n$ and $y_0',\ldots,y_n'$ are related by $y_i=\kappa_i y_i'$ for some $\kappa_i\in\mathcal{O}_Y^*(Y)$. The corresponding $\boldsymbol{\Upsilon}(A_{n-1})$-collections $\mathscr{L}$ and $\mathscr{L}'$ are connected by the isomorphism $\mathscr{L}'\to\mathscr{L}$ consisting of isomorphisms $\mathcal{O}_Y=\mathscr{L}_{\varrho_i}\to\mathscr{L}_{\varrho_i}'=\mathcal{O}_Y$, $\mathcal{O}_Y=\mathscr{L}_{\tau_i}\to\mathscr{L}_{\tau_i}'=\mathcal{O}_Y$ given by a collection $\kappa_1,\ldots,\kappa_{n-1},\lambda_1,\ldots,\lambda_{n-1}\in\mathcal{O}_Y^*(Y)$ as in remark 2.5.

For a morphism $\mathscr{C}'\to\mathscr{C}$ of degree-$n$-pointed chains over $f\colon Y'\to Y$, i.e. a cartesian diagram consisting of $f$, a morphism $F\colon C'\to C$ that maps $s_-',s_+',S'$ to $s_-,s_+,S$ and $\pi\colon C\to Y$, $\pi'\colon C'\to Y'$, we have a morphism of the corresponding $\boldsymbol{\Upsilon}(A_{n-1})$-collections $\mathscr{L}'\to\mathscr{L}$ that arises as above (we have given trivialisations $\mathscr{L}_{\varrho_i}',\mathscr{L}_{\tau_i}'=\mathcal{O}_{Y'}$ and $f^*\mathscr{L}_{\varrho_i},f^*\mathscr{L}_{\tau_i}=f^*\mathcal{O}_Y\cong\mathcal{O}_{Y'}$) by comparing the chosen bases $f^*y_0,\ldots,f^*y_n$ and $y_0',\ldots,y_n'$ of $f^*\pi_*\mathcal{O}_C(S)\cong\pi_*'F^*\mathcal{O}_C(S)\cong\pi_*'\mathcal{O}_{C'}(S')$.

One checks that this defines a functor $\Phi\colon\overline{\mathcal{L}}_n\to\mathcal{C}_{\boldsymbol{\Upsilon}(A_{n-1})}$. The functor $\Phi$ is base-preserving and sends cartesian arrows to cartesian arrows.

We also construct a functor in the opposite direction.

**Construction 3.6.** Let $\mathscr{L}=((\mathscr{L}_{\varrho_i},a_i)_i,(\mathscr{L}_{\tau_i},b_i)_i,(c_i)_i)$ be a $\boldsymbol{\Upsilon}(A_{n-1})$-collection on an affine scheme $Y$. We choose trivialisations $\mathscr{L}_{\varrho_i},\mathscr{L}_{\tau_i}\cong\mathcal{O}_Y$ such that the isomorphisms $c_i\colon\mathscr{L}_{\tau_i}\otimes\mathscr{L}_{\varrho_{i-1}}\otimes\mathscr{L}_{\varrho_i}^{\otimes\text{-}2}\otimes\mathscr{L}_{\varrho_{i+1}}\to\mathcal{O}_Y$ for $i=1,\ldots,n-1$ (omit $\mathscr{L}_{\varrho_0},\mathscr{L}_{\varrho_n}$) are the identities on $\mathcal{O}_Y$. Let $C$ be the closed subscheme of $\mathbb{P}^n_Y$ given by the equations $y_iy_{j+1}=b_{i+1}\cdots b_jy_{i+1}y_j$ for $0\le i<j\le n-1$, where $y_0,\ldots,y_n$ are homogeneous coordinates of $\mathbb{P}^n_Y$ and $b_1,\ldots,b_{n-1}$ are considered as regular functions on $Y$ via the isomorphisms $\mathscr{L}_{\tau_i}\cong\mathcal{O}_Y$, and let $\pi\colon C\to Y$ be induced by $\mathbb{P}^n_Y\to Y$. By construction, the subscheme $C\subseteq\mathbb{P}^n_Y$ is isomorphic to $\mathrm{Proj}_Y\mathscr{S}$ where $\mathscr{S}$ is the graded algebra $\mathcal{O}_Y[y_0,\ldots,y_n]\,/\,\langle y_iy_{j+1}=b_{i+1}\cdots b_jy_{i+1}y_j;i<j\rangle$. The morphism $\pi\colon C\to Y$ is flat since each graded piece of $\mathscr{S}$ is locally free ([EGA, III, (7.9.14)]). Indeed, we have $\mathscr{S}_k\cong\mathcal{O}_Y^{\oplus kn+1}$ with basis $y_n^k$ and $y_i^{k-l}y_{i+1}^l$ for $i=0,\ldots,n-1$ and $l=0,\ldots,k-1$. Let $s_-,s_+$ be the sections $(1:0:\ldots:0)$, $(0:\ldots:0:1)$ with respect to the co-ordinates $y_0,\ldots,y_n$ and let $S\subset C$ be the subscheme given by the additional equation $y_n+a_{n-1}y_{n-1}+\ldots+a_1y_1+y_0=0$, where again $a_1,\ldots,a_{n-1}$ are considered as regular functions via $\mathscr{L}_{\varrho_i}\cong\mathcal{O}_Y$. This defines a degree-$n$-pointed chain of projective lines $\mathscr{C}=(C\to Y,s_-,s_+,S)$.

For a morphism $\mathscr{L}'\to\mathscr{L}$ of $\boldsymbol{\Upsilon}(A_{n-1})$-collections over $f\colon Y'\to Y$ we have a morphism $\mathscr{C}'\to\mathscr{C}$ of degree-$n$-pointed chains. The curve $\mathscr{C}$ associated to $\mathscr{L}$ consists of subschemes of $\mathbb{P}^n_Y$ defined with respect to homogeneous coordinates $y_0,\ldots,y_n$ in terms of the functions $a_i,b_i\in\mathcal{O}_Y(Y)$ after choice of isomorphisms $\mathscr{L}_{\varrho_i},\mathscr{L}_{\tau_i}\cong\mathcal{O}_Y$, and its pullback $f^*\mathscr{C}$, coming with an embedding into $\mathbb{P}^n_Y\times_YY'\cong\mathbb{P}^n_{Y'}$, coincides with the subschemes of $\mathbb{P}^n_{Y'}$ defined with respect to homogeneous coordinates $f^*y_i$ in terms of the functions $f^*a_i,f^*b_i\in\mathcal{O}_{Y'}(Y')$, making use of the natural isomorphism $f^*\mathcal{O}_Y\cong\mathcal{O}_{Y'}$. The curve $\mathscr{C}'$ associated to $\mathscr{L}'$ consists of subschemes of $\mathbb{P}^n_{Y'}$

defined with respect to homogeneous coordinates $y'_0, \dots, y'_n$ in terms of the functions $a'_i, b'_i \in \mathcal{O}_{Y'}(Y')$ after choice of isomorphisms $\mathscr{L}'_{\varrho_i}, \mathscr{L}'_{\tau_i} \cong \mathcal{O}_{Y'}$. The composition of isomorphisms $\mathcal{O}_{Y'} \cong f^*\mathcal{O}_Y \cong f^*\mathscr{L}_{\varrho_i} \to \mathscr{L}'_{\varrho_i} \cong \mathcal{O}_{Y'}$ is given by multiplication by some $\kappa_i \in \mathcal{O}^*_{Y'}(Y')$, so we have $\kappa_i f^* a_i = a'_i$. The isomorphism $\mathbb{P}^n_{Y'} \to \mathbb{P}^n_{Y'} \cong \mathbb{P}^n_Y \times_Y Y'$ such that $\kappa_i y'_i$ corresponds to $f^* y_i$ induces a morphism $\mathscr{C}' \to f^*\mathscr{C}$. Its composition with $f^*\mathscr{C} \to \mathscr{C}$ (induced by $\mathbb{P}^n_Y \times_Y Y' \to \mathbb{P}^{n+1}_Y$) gives the morphism $\mathscr{C}' \to \mathscr{C}$.

This defines a functor $\Psi\colon \mathcal{C}_{\Upsilon(A_{n-1})} \to \overline{\mathcal{L}}_n$ which is base-preserving and sends cartesian arrows to cartesian arrows.

These two functors give the equivalence of fibred categories stated in the theorem.

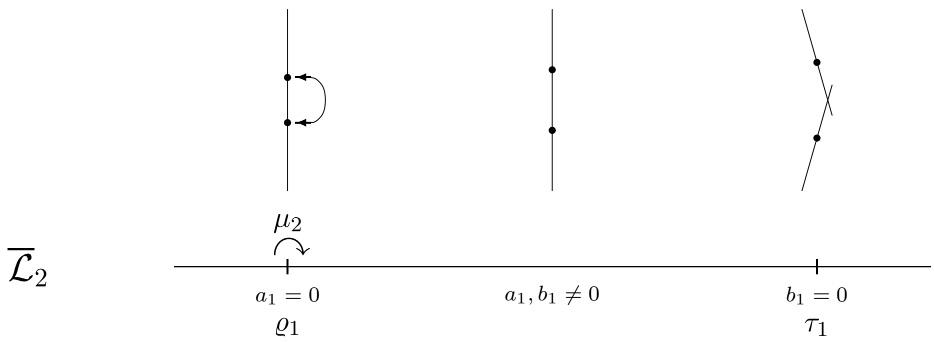
*Proof of theorem 3.1.* We show that the fibred categories $\overline{\mathcal{L}}_n$ and $\mathcal{C}_{\Upsilon(A_{n-1})}$ are equivalent using the functors $\Psi\colon \mathcal{C}_{\Upsilon(A_{n-1})} \to \overline{\mathcal{L}}_n$ and $\Phi\colon \overline{\mathcal{L}}_n \to \mathcal{C}_{\Upsilon(A_{n-1})}$.

We have an isomorphism of functors $\Psi \circ \Phi \cong \mathrm{Id}$, where for an object $\mathscr{C} = (C \to Y, s_-, s_+, S)$ in $\overline{\mathcal{L}}_n$ the isomorphism $\mathscr{C} \to \Psi\Phi(\mathscr{C})$ is given as the embedding $C \to \mathbb{P}_Y(\pi_*\mathcal{O}_C(S))$ determined by $\mathcal{O}_C(S)$ together with an isomorphism $\mathbb{P}_Y(\pi_*\mathcal{O}_C(S)) \cong \mathbb{P}^n_Y$ prescribed by choice of trivialisations of the line bundles of the data $\Phi\mathscr{C}$. More precisely, the data $\Phi\mathscr{C}$ consists of $(\mathscr{L}_{\varrho_i} = \mathcal{O}_Y, a_i)$, $(\mathscr{L}_{\tau_i} = \mathcal{O}_Y, b_i)$ and the functions $a_i, b_i$ determine the embedded object in $\mathbb{P}_Y(\pi_*\mathcal{O}_C(S))$ with respect to a basis $y_0, \dots, y_n$, see proposition 3.4. Applying the functor $\Psi$, we choose isomorphisms $\mathscr{L}_{\varrho_i}, \mathscr{L}_{\tau_i} \to \mathcal{O}_Y$ as in construction 3.6 giving rise to functions $\tilde{a}_i, \tilde{b}_i$, and these define the object $\Psi\Phi\mathscr{C}$ in $\mathbb{P}^n_Y$ with homogeneous coordinates $\tilde{y}_i$. The isomorphisms $\mathcal{O}_Y = \mathscr{L}_{\varrho_i} \to \mathcal{O}_Y$ are multiplications with some $\kappa_i \in \mathcal{O}^*_Y(Y)$, and the isomorphism $\mathbb{P}_Y(\pi_*\mathcal{O}_C(S)) \cong \mathbb{P}^n_Y$ such that coordinates $y_i$ correspond to $\kappa_i \tilde{y}_i$ induces an isomorphism between the object $\mathscr{C}$ embedded in $\mathbb{P}_Y(\pi_*\mathcal{O}_C(S))$ and $\Psi\Phi\mathscr{C}$ embedded in $\mathbb{P}^n_Y$. One checks that these isomorphisms form an isomorphism of functors.

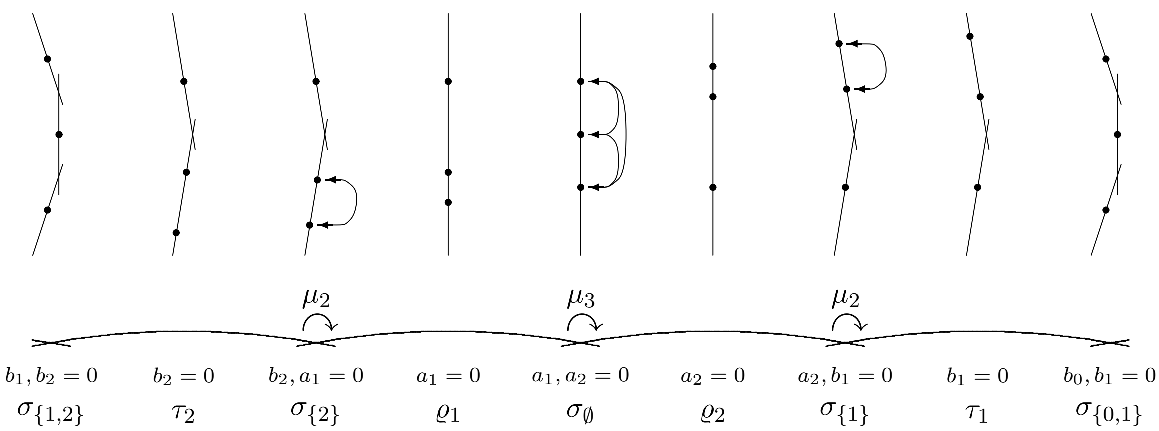
Starting with data $\mathscr{L}$, we define an object $\Psi\mathscr{L} = (C \to Y, s_-, s_+, S)$ in $\mathbb{P}^n_Y$ with homogeneous coordinates $y_0, \dots, y_n$ using functions $a_i, b_i$ obtained by choice of an isomorphism $\mathscr{L} \cong ((\mathcal{O}_Y, a_i)_i, (\mathcal{O}_Y, b_i)_i, (id)_i))$, and then extract data $\Phi\Psi\mathscr{L}$ consisting of line bundles with sections $(\tilde{\mathscr{L}}_{\varrho_i} = \mathcal{O}_Y, \tilde{a}_i)$, $(\tilde{\mathscr{L}}_{\tau_i} = \mathcal{O}_Y, \tilde{b}_i)$ after choice of a basis $\tilde{y}_0, \dots, \tilde{y}_n$ of $\pi_*(\mathcal{O}_C(S))$ as in construction 3.5. Let $\bar{S} \subset \mathbb{P}^n_Y$ be the hyperplane determined by the equation $\sum_i a_i y_i = 0$, putting $a_0, a_{n+1} = 1$ (its restriction to the curve $C \subset \mathbb{P}^n_Y$ is the subscheme $S$). The isomorphism $\mathbb{P}^n_Y \to \mathbb{P}_Y(\pi_*\mathcal{O}_{\mathbb{P}^n_Y}(\bar{S})) = \mathbb{P}_Y(\pi_*\mathcal{O}_C(S))$ induces the embedding $C \to \mathbb{P}_Y(\pi_*\mathcal{O}_C(S))$. Since the coordinates $y_i$ also satisfy the conditions of proposition 3.4, $y_i$ corresponds to $\kappa_i \tilde{y}_i$ for some $\kappa_i \in \mathcal{O}^*_Y(Y)$. Thus the functions $a_i, b_i$ and $\tilde{a}_i, \tilde{b}_i$ are related via $\kappa_i a_i = \tilde{a}_i$, $\lambda_i b_i = \tilde{b}_i$ where $\lambda_i = \kappa_i^2/(\kappa_{i-1}\kappa_{i+1})$ (set $\kappa_0 = \kappa_n = 1$), and the data $\Phi\Psi\mathscr{L}$ is related to the original data via an isomorphism $\Phi\Psi\mathscr{L} \to \mathscr{L}$ consisting of isomorphisms $\mathscr{L}_{\varrho_i} \cong \mathcal{O}_Y \to \mathcal{O}_Y$, $\mathscr{L}_{\tau_i} \cong \mathcal{O}_Y \to \mathcal{O}_Y$ composed of the chosen trivialisations and multiplication by $\kappa_i, \lambda_i$. One verifies that this gives an isomorphism of functors $\Phi \circ \Psi \cong \mathrm{Id}$. $\square$

**Corollary 3.7.** *The coarse moduli space of $\overline{\mathcal{L}}_n$, which coincides with the quotient $\overline{L}_n/S_n$, is isomorphic to the toric variety $Y(A_{n-1})$ corresponding to the simplicial fan $\Upsilon(A_{n-1})$ underlying the stacky fan $\boldsymbol{\Upsilon}(A_{n-1})$.*

**Example 3.8.** In the case $n=2$ we have the isomorphism $\overline{\mathcal{L}}_2 \cong \mathcal{Y}(A_1)$. The stacky fan of $\mathcal{Y}(A_1)$ was pictured in example 2.2. We have the following types of pointed chains over $\overline{\mathcal{L}}_2 \cong \mathcal{Y}(A_1)$ (cf. also example 1.8):




**Example 3.9.** In the case $n=3$ we have the isomorphism $\overline{\mathcal{L}}_3 \cong \mathcal{Y}(A_2)$. The stacky fan of $\mathcal{Y}(A_2)$ appeared in example 2.3. Here we picture the types of pointed chains over the torus invariant divisors of the moduli stack $\overline{\mathcal{L}}_3$.



## 4. The functor of $X(A_{n-1})$, Losev-Manin moduli spaces and the morphism to $\overline{\mathcal{L}}_n$

We start by comparing three descriptions of the functor of the toric variety associated with root systems of type $A$. We use notations as in [BB11a, Section 2.1], in particular we have the lattice $M(A_{n-1}) = \langle u_i - u_j : i, j \in \{1, \dots, n\}\rangle \subset \bigoplus_{i=1}^n \mathbb{Z}u_i$, generated by the roots $\beta_{ij} = u_i - u_j$ and forming the character lattice for the toric variety $X(A_{n-1})$. Its dual $N(A_{n-1}) = \bigoplus_{i=1}^n \mathbb{Z}v_i / \sum_i v_i$, where $(u_i)_i$ and $(v_i)_i$ are dual bases, is the lattice for the fan $\Sigma(A_{n-1})$ of $X(A_{n-1})$.

The functor of the toric variety $X(A_{n-1})$ was described in [BB11a] in terms of $A_{n-1}$-data, i.e. families $(\mathscr{L}_{\{\pm\beta_{ij}\}}, \{t_{\beta_{ij}}, t_{-\beta_{ij}}\})_{\{\pm\beta_{ij}\}}$ of line bundles with two generating sections that satisfy $t_\alpha t_\beta t_{-\gamma} = t_{-\alpha} t_{-\beta} t_\gamma$ if $\gamma = \alpha + \beta$, up to isomorphism of line bundles with a pair of sections. With pull-back of line bundles and its sections we have the functor $F_{A_{n-1}}$ of $A_{n-1}$-data, see [BB11a, Def. 1.17].

On $X(A_{n-1})$ we have the universal $A_{n-1}$-data, which can be defined using the morphisms $\varphi_{\{\pm\beta_{ij}\}} \colon X(A_{n-1}) \to \mathbb{P}^1$ induced by pairs of opposite roots $\{\pm\beta_{ij}\}$ in $A_{n-1}$ (see [BB11a, Ex. 1.5 and 1.13]). We have homogeneous coordinates $z_{\beta_{ij}}, z_{-\beta_{ij}} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $x^{\beta_{ij}} = \varphi^*_{\{\pm\beta_{ij}\}}(z_{\beta_{ij}}/z_{-\beta_{ij}})$, where $x^u$ for $u \in M(A_{n-1})$ is the rational function corresponding to an element of the root lattice. Let $\mathscr{L}_{\{\pm\beta_{ij}\}} = \varphi^*_{\{\pm\beta_{ij}\}}\mathcal{O}_{\mathbb{P}^1}(1)$ and $t_{\beta_{ij}}, t_{-\beta_{ij}}$ be the pull-back of $z_{\beta_{ij}}, z_{-\beta_{ij}}$.

By [BB11a, Thm. 1.20] the toric variety $X(A_{n-1})$ together with the universal $A_{n-1}$-data represents the functor $F_{A_{n-1}}$.

We can also apply the description of the functor of a smooth toric variety of Cox [Co95b] to $X(A_{n-1})$. The fan $\Sigma(A_{n-1})$ gives rise to the notion of a $\Sigma(A_{n-1})$-collection $((\mathscr{L}_I, w_I)_I, (c_{ij})_{ij})$ on a scheme $Y$, consisting of line bundles on $Y$ with a global section $(\mathscr{L}_I, w_I)$ for $\emptyset \neq I \subsetneq \{1, \dots, n\}$ and isomorphisms $c_{ij} \colon \big(\bigotimes_{i\in I, j\notin I} \mathscr{L}_I\big) \otimes \big(\bigotimes_{i\notin I, j\in I} \mathscr{L}_I^{\otimes -1}\big) \to \mathcal{O}_Y$ for $i, j \in \{1, \dots, n\}$, $i \neq j$, such that identifications of the form $c_{ij} \otimes c_{jk} = c_{ik}$ hold. These data have to satisfy the nondegeneracy condition that for any point $y \in Y$ there are sets $I_1 \subset \ldots \subset I_{n-1} \subset \{1, \dots, n\}$ with $|I_i| = i$ such that $w_I(y) \neq 0$ if $I \neq I_1, \dots, I_{n-1}$. We denote the functor of $\Sigma(A_{n-1})$-collections by $C_{\Sigma(A_{n-1})}$.

On $X(A_{n-1})$ we have the universal $\Sigma(A_{n-1})$-collection given by the line bundles $\mathscr{L}_I = \mathcal{O}_{X(A_{n-1})}(D_I)$, where $D_I$ is the torus invariant prime divisor corresponding to the ray generated by $\sum_{i\in I} v_i$, with the section $w_I$ arising as the image of the 1-section under the natural inclusion $\mathcal{O}_{X(A_{n-1})} \to \mathcal{O}_{X(A_{n-1})}(D_I)$ and the isomorphisms $c_{ij} \colon \mathcal{O}_{X(A_{n-1})}(\sum_{i\in I, j\notin I} D_I - \sum_{i\notin I, j\in I} D_I) \to \mathcal{O}_{X(A_{n-1})}$ induced by multiplication with the rational functions $x^{\beta_{ij}}$ on $X(A_{n-1})$.

By [Co95b] the toric variety $X(A_{n-1})$ together with the universal $\Sigma(A_{n-1})$-collection $((\mathcal{O}_{X(A_{n-1})}(D_I), w_I)_I, (c_{ij})_{ij})$ represents the functor $C_{\Sigma(A_{n-1})}$.

As both functors $C_{\Sigma(A_{n-1})}$ and $F_{A_{n-1}}$ are isomorphic to the functor of the toric variety $X(A_{n-1})$, we have an isomorphism of functors $C_{\Sigma(A_{n-1})} \to F_{A_{n-1}}$, which we describe explicitely.

**Proposition 4.1.** *By the following procedure we can construct $A_{n-1}$-data $(\mathscr{L}_{\{\pm\beta_{ij}\}},\{t_{\beta_{ij}},t_{-\beta_{ij}}\})_{\{\pm\beta_{ij}\}}$ out of a $\Sigma(A_{n-1})$-collection $((\mathscr{L}_I,w_I)_I,(c_{ij})_{ij})$ over a scheme $Y$: for a pair of opposite roots $\pm\beta_{ij}$ in $A_{n-1}$ we have isomorphisms $\bigotimes_{i\in I,j\notin I}\mathscr{L}_I \overset{\sim}{\longleftrightarrow} \bigotimes_{i\notin I,j\in I}\mathscr{L}_I$ of line bundles on $Y$ defined by $c_{ij},c_{ji}$ inverse to each other, and we let $\mathscr{L}_{\{\pm\beta_{ij}\}}$ be a line bundles in the same isomorphism class, with the sections $t_{\beta_{ij}},t_{-\beta_{ij}}$ defined as the images of $\prod_{i\in I,j\notin I}w_I,\prod_{i\notin I,j\in I}w_I$ in $\mathscr{L}_{\{\pm\beta_{ij}\}}$ under isomorphisms compatible with the above. This construction defines an isomorphism of functors $C_{\Sigma(A_{n-1})}\to F_{A_{n-1}}$, mapping the universal $\Sigma(A_{n-1})$-collection to the universal $A_{n-1}$-data.*

*Proof.* This construction defines a morphism of functors $C_{\Sigma(A_{n-1})}\to F_{A_{n-1}}$, in particular the requirement that the two sections $t_{\pm\beta_{ij}}$ as defined in the construction generate the line bundle $\mathscr{L}_{\{\pm\beta_{ij}\}}$ follows from the nondegeneracy condition of $\Sigma(A_{n-1})$-data. One can show that this morphism of functors is an isomorphism by showing that it coincides with the composition of isomorphisms $C_{\Sigma(A_{n-1})}\to \mathrm{Mor}(\,\cdot\,,X(A_{n-1}))\to F_{A_{n-1}}$. This follows from the fact that the universal $\Sigma(A_{n-1})$-collection is mapped to the universal $A_{n-1}$-data, which is easy to verify. □

Considering the universal data on $X(A_{n-1})$, we have isomorphisms $\bigotimes_{i\in I}\mathscr{L}_I \overset{\sim}{\to} \bigotimes_{j\in I}\mathscr{L}_I$ via multiplication by the rational function $x^{\beta_{ij}}$. For any chosen $j\in\{1,\dots,n\}$ we may define $x_1,\dots,x_n\in H^0(X(A_{n-1}),\bigotimes_{j\in I}\mathscr{L}_I)$ as images of the sections $\prod_{1\in I}w_I,\dots,\prod_{n\in I}w_I$ under these isomorphisms. We then have $x^{\beta_{ij}}=x_i/x_j$.

**Definition 4.2.** Given an ordering $i_1,\dots,i_n$ of the set $\{1,\dots,n\}$, we define line bundles $\mathscr{L}_1,\dots,\mathscr{L}_{n-1}$ on $X(A_{n-1})$ and sections $x_J$ of $\mathscr{L}_{|J|}$. Let

$$\mathscr{L}_j=\bigotimes\nolimits_I \mathscr{L}_I^{\otimes(|\{i_1,\dots,i_j\}\cap I|-\max\{0,|I|+j-n\})}$$

be defined in terms of the universal $\Sigma(A_{n-1})$-collection. The line bundles $\bigotimes_I \mathscr{L}_I^{\otimes(|J\cap I|-\max\{0,|I|+j-n\})}$ for any $J\subset\{1,\dots,n\}$ of cardinality $j$ are isomorphic to $\mathscr{L}_j$ via multiplication by $\prod_{i\in J}x_i/\prod_{k=1}^j x_{i_k}$ (can also be expressed in terms of the isomorphisms $c_{ij}$ being part of the universal $\Sigma(A_{n-1})$-collection). We define $x_J\in H^0(X(A_{n-1}),\mathscr{L}_j)$ as the image of $\prod_I w_I^{|J\cap I|-\max\{0,|I|+j-n\}}$ under this isomorphism.

For these sections $x_J\in H^0(X(A_{n-1}),\mathscr{L}_{|J|})$ we have equations of rational functions $\prod_{j\in J}x_j/\prod_{j\in J'}x_j=x_J/x_{J'}=\prod_{j\in J'}x_{\{1,\dots,n\}\setminus\{j\}}/\prod_{j\in J}x_{\{1,\dots,n\}\setminus\{j\}}$.

**Remark 4.3.** The line bundle $\mathscr{L}_j$ was defined as $\mathcal{O}_{X(A_{n-1})}(D)$ in terms of the divisor $D=\sum_I d_ID_i$, where $d_I=|\{i_1,\dots,i_j\}\cap I|-\max\{0,|I|+j-n\}$, which corresponds to the lattice polytope

$$\begin{aligned}\Delta_j(A_{n-1}) &= \textstyle\bigcap_I\{\,u\in M(A_{n-1})_{\mathbb{Q}}\,:\,\sum_{i\in I}v_i(u)\ge -d_I\,\}\\ &= \mathrm{conv}\,\{\,\textstyle\sum_{i\in J}u_i-\sum_{k=1}^j u_{i_k}\,:\,|J|=j\,\}\end{aligned}$$

in $M(A_{n-1})_{\mathbb{Q}}$; the elements $x_J\in H^0(X(A_{n-1}),\mathscr{L}_j)$ for $|J|=j$ form a basis of global sections. Different choices of the ordering fixed in the definition give rise to translated polytopes.

The line bundle $\mathscr{L}_1$ with its basis of global sections $x_1, \ldots, x_n$ defines a morphism $X(A_{n-1}) \to \mathbb{P}^{n-1}$. This morphism is a composition of toric blow-ups as described in [Ka93, (4.3.13)]. It maps the divisors $D_{\{i\}}$ of $X(A_{n-1})$ to the torus invariant prime divisor $D_i = \{x_i = 0\}$ of $\mathbb{P}^{n-1}$, and more generally $D_I$ to $\bigcap_{i \in I} D_i$. Similarly, we have a morphism $X(A_{n-1}) \to \mathbb{P}^{n-1}$ defined by the line bundle $\mathscr{L}_{n-1}$ mapping the divisor $D_{\{1,\ldots,n\}\setminus\{i\}}$ of $X(A_{n-1})$ to the torus invariant prime divisor $D_i^* = \{x_{\{1,\ldots,n\}\setminus\{i\}} = 0\}$ of $\mathbb{P}^{n-1}$, and more generally $D_{\{1,\ldots,n\}\setminus I}$ to $\bigcap_{i \in I} D_i^*$, see also [Ka93, (4.3.14)]. We also consider the morphisms defined by the other line bundles $\mathscr{L}_j$:

**Proposition 4.4.** *The line bundle $\mathscr{L}_j$ is generated by the basis of global sections $(x_J)_{|J|=j}$, it determines a projective toric morphism*

$$X(A_{n-1}) \ \to \ \mathbb{P}(\langle x_J : |J| = j \rangle) \cong \mathbb{P}^{\binom{n}{j}-1}$$

*which is birational onto its image. Together, these morphisms form a closed embedding*

$$X(A_{n-1}) \ \to \ \textstyle\prod_{j=1}^{n-1} \mathbb{P}(\langle x_J : |J| = j \rangle) \cong \prod_{j=1}^{n-1} \mathbb{P}^{\binom{n}{j}-1}. \tag{5}$$

*The subscheme $X(A_{n-1})$ in this product is defined by homogeneous equations*

$$\textstyle\prod_{i=1}^{l} x_{J_i} \ = \ \prod_{i=1}^{l} x_{J_i'} \tag{6}$$

*where $\emptyset \neq J_i, J_i' \subsetneq \{1, \ldots, n\}$ such that $|J_i| = |J_i'|$ and the equation for the characteristic functions $\sum_i \chi_{J_i} = \sum_i \chi_{J_i'}$ is satisfied.*

*Proof.* That $\mathscr{L}_j$ is generated by global sections $(x_J)_{|J|=j}$ and determines a projective toric morphism follows from the fact that $\mathscr{L}_j$ can be reconstructed from the polytope $\Delta_j(A_{n-1})$, see remark 4.3. This morphism is birational onto its image since the polytope is full-dimensional. We describe these morphisms in terms of the corresponding maps of fans.

For the toric variety $\mathbb{P}(\langle x_J : |J| = j \rangle)$ we have the character lattice $M(A_{n-1})_j \subset \bigoplus_{|J|=j} \mathbb{Z} u_J$ generated by differences $u_J - u_{J'}$ and the dual lattice $N(A_{n-1})_j = (\bigoplus_{|J|=j} \mathbb{Z} v_J)/(\sum_J v_J)$. The fan of $\mathbb{P}(\langle x_J : |J| = j \rangle)$ has the one-dimensional cones generated by the $v_J$. The morphism $X(A_{n-1}) \to \mathbb{P}(\langle x_J : |J| = j \rangle)$ is determined by the map of lattices $M(A_{n-1})_j \to M(A_{n-1})$, $u_J \mapsto \sum_{i \in J} u_i$, or dually $N(A_{n-1}) \to N(A_{n-1})_j$, $v_i \mapsto \sum_{i \in J} v_J$, which defines a map of fans.

The product of these morphisms is given by the map of lattices $\bigoplus_{j=1}^{n} M(A_{n-1})_j \to M(A_{n-1})$ with kernel generated by elements of the form $\sum_i n_{J_i} - \sum_i n_{J_i'}$ such that $|J_i| = |J_i'|$ and $\sum_i \chi_{J_i} = \sum_i \chi_{J_i'}$. This gives rise to the homogeneous equations.

The maximal cones of the fan of $\mathbb{P}(\langle x_J : |J| = j \rangle)$ are the cones $\sigma_J$ generated by $\{v_{J'} \mid J' \neq J, |J'| = j\}$. For an ordering $i_1, \ldots, i_n$ of $\{1, \ldots, n\}$ the preimage of the maximal cone $\sigma_{\{i_n\}} \times \sigma_{\{i_n, i_{n-1}\}} \times \ldots \times \sigma_{\{i_n, \ldots, i_2\}}$ of the product fan is the maximal cone of $\Sigma(A_{n-1})$ generated by $v_{i_1}, v_{i_1} + v_{i_2}, \ldots, v_{i_1} + \ldots + v_{i_{n-1}}$, thus the open sets corresponding to maximal cones of this form cover the image of $X(A_{n-1})$. The corresponding maps of coordinate algebras are surjective, so the morphism is a closed embedding. □

**Remark 4.5.** The $(n-1)$-dimensional permutohedron, usually defined in an $n$-dimensional vector space as convex hull of the orbit of $(n, n-1, \ldots, 1)$ under the action of the symmetric group $S_n$ permuting the given basis (cf. for example [Ka93, (4.3.10)]), can be considered as a lattice polytope in $M(A_{n-1})_{\mathbb{Q}} \subset \mathbb{Q}^n$ after a translation moving one of its vertices, specified by fixing an ordering $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$, to the origin:

$$\Delta(A_{n-1}) = \operatorname{conv}\left\{ \sum_{k=1}^{n-1}(n-k)u_{\sigma(k)} - \sum_{k=1}^{n-1}(n-k)u_{i_k} : \sigma \in S_n \right\}.$$

We have Minkowski sum decompositions of the permutohedron, first

$$\Delta(A_{n-1}) = \sum_{k<j} l_{i_j i_k}$$

into line segments $l_{ij} = \{r \cdot \beta_{ij} \mid 0 \le r \le 1\}$ corresponding the line bundles $\mathscr{L}_{\{\pm\beta_{ij}\}}$ forming the universal $A_{n-1}$-data (choosing $\mathcal{O}_{X(A_{n-1})}(\sum_{i_k \in I, i_j \notin I} D_I)$ in the isomorphism class of $\mathscr{L}_{\{\pm\beta_{i_j i_k}\}}$ if $k < j$), and second

$$\Delta(A_{n-1}) = \Delta_1(A_{n-1}) + \ldots + \Delta_{n-1}(A_{n-1})$$

into the polytopes corresponding to the line bundles $\mathscr{L}_j$.

**Remark 4.6.** The closed embedding (5) together with the functor of projective spaces gives another description of the functor of the toric variety $X(A_{n-1})$. We have a contravariant functor on the category of schemes: its data on a scheme $Y$ are line bundles with generating sections $(\mathscr{L}_j, (x_J)_{|J|=j})_{j=1,\ldots,n-1}$ up to isomorphism such that the sections satisfy the relations (6), and for morphisms of schemes we have the pull-back of line bundles with sections. We call the data on $X(A_{n-1})$ introduced in definition 4.2 the universal data on $X(A_{n-1})$. Then, the toric variety $X(A_{n-1})$ together with the universal data represents this functor. Further, the method of definition 4.2 applied to $\Sigma(A_{n-1})$-data over arbitrary schemes gives a morphism from $C_{\Sigma(A_{n-1})}$ to this functor, mapping the universal $\Sigma(A_{n-1})$-collection to the universal data. As in the proof of proposition 4.1 this implies that we have an isomorphism of functors.

The following observation can be directly calculated from the definition of the line bundles $\mathscr{L}_j$.

**Lemma 4.7.** *We have isomorphisms*

$$\mathscr{L}_{j-1}^{\otimes -1} \otimes \mathscr{L}_j^{\otimes 2} \otimes \mathscr{L}_{j+1}^{\otimes -1} \;\cong\; \bigotimes_{|J|=n-j} \mathscr{L}_J \tag{7}$$

*where we set* $\mathscr{L}_0 = \mathscr{L}_n = \mathcal{O}_{X(A_{n-1})}$.

**Definition 4.8.** We define the divisors $C_1, \ldots, C_{n-1}$ and $D_1, \ldots, D_{n-1}$ on $X(A_{n-1})$. Let $D_j = \sum_{|J|=n-j} D_J$ and let $C_j$ be the zero divisor of the section $\sum_{|J|=j} x_J$ of the line bundle $\mathscr{L}_j$.

**Remark 4.9.** We may write the isomorphism (7) as linear equivalence of divisors

$$2C_j - C_{j-1} - C_{j+1} \;\sim\; D_j\,.$$

The rational function

$$\frac{(\sum_{|I|=j-1} x_I)(\sum_{|I|=j+1} x_I)}{(\sum_{|I|=j} x_I)^2} \tag{8}$$

has divisor $D_j + C_{j-1} + C_{j+1} - 2C_j$.

**Lemma 4.10.** *For $j = 1, \ldots, n-1$ we have $C_j \cap D_j = \emptyset$.*

*Proof.* Can easily be checked locally using the covering of the following remark. $\square$

**Remark 4.11.** Given a $\Sigma(A_{n-1})$-collection $((\mathscr{L}_I, w_I)_I, (c_{ij})_{ij})$ on a scheme $Y$, by nondegeneracy we have the following covering of $Y$ by open subschemes: for a permutation $\sigma \in S_n$ set $\mathcal{I}_\sigma = \big\{\{\sigma(n)\}, \{\sigma(n), \sigma(n-1)\}, \ldots, \{\sigma(n), \ldots, \sigma(2)\}\big\}$ and let $W_\sigma$ be the open subscheme of $Y$ where $w_I \neq 0$ for $I \notin \mathcal{I}_\sigma$.

In the case of the universal $\Sigma(A_{n-1})$-collection on $X(A_{n-1})$ the subscheme $W_\sigma \subset X(A_{n-1})$ corresponds to the maximal cone $\langle v_{\sigma(n)}, \ldots, v_{\sigma(n)} + \ldots + v_{\sigma(2)} \rangle \subset N(A_{n-1})_\mathbb{Q}$ dual to the cone generated by the simple roots $u_{\sigma(n)} - u_{\sigma(n-1)}, \ldots, u_{\sigma(2)} - u_{\sigma(1)}$ and has as coordinate algebra the polynomial ring generated by $\frac{x_{\sigma(n)}}{x_{\sigma(n-1)}}, \ldots, \frac{x_{\sigma(2)}}{x_{\sigma(1)}}$.

By [BB11a, Thm. 3.19] there is an isomorphism between the functor $F_{A_{n-1}}$ and the moduli functor of $n$-pointed chains of projective lines $\overline{L}_n$ mapping the universal $A_{n-1}$-data to the universal $n$-pointed chain $(X(A_n) \to X(A_{n-1}), s_-, s_+, s_1, \ldots, s_n)$ defined in [BB11a, Con. 3.6]. This means that the toric variety $X(A_{n-1})$ coincides with the Losev-Manin moduli space $\overline{L}_n$ (we use the same symbol for the functor and the moduli space). The construction uses an embedding of $n$-pointed chains into $(\mathbb{P}^1)^n$.

This also implies that there is an isomorphism between the functor $C_{\Sigma(A_{n-1})}$ and the moduli functor $\overline{L}_n$ compatible with the other isomorphisms of functors. We make this isomorphism explicit using an embedding of $n$-pointed chains into $\mathbb{P}^n$.

**Construction 4.12.** Let $((\mathscr{L}_I, w_I)_I, (c_{ij})_{ij})$ be a $\Sigma(A_{n-1})$-collection over a scheme $Y$. We construct an $n$-pointed chain of projective lines $(C \to Y, s_-, s_+, s_1, \ldots, s_n)$ using the covering of $Y$ by $(W_\sigma)_{\sigma \in S_n}$ (see remark 4.11).

For $\sigma \in S_n$ the restricted $\Sigma(A_{n-1})$-collection $((\mathscr{L}_I|_{W_\sigma}, w_I|_{W_\sigma})_I, (c_{ij}|_{W_\sigma})_{ij})$ is isomorphic to a $\Sigma(A_{n-1})$-collection $((\mathscr{L}_I^\sigma, w_I^\sigma)_I, (c_{ij}^\sigma)_{ij})$ on $W_\sigma$ with the property $(\mathscr{L}_I^\sigma, w_I^\sigma) = (\mathcal{O}_{W_\sigma}, 1)$ for $I \notin \mathcal{I}_\sigma$, and for $i = 1, \ldots, n-1$ we have isomorphisms $c^\sigma_{\sigma(i+1),\sigma(i)} \colon \mathscr{L}^\sigma_{\{\sigma(n),\ldots,\sigma(i+1)\}} \to \mathcal{O}_{W_\sigma}$. Let $w_i^\sigma \in \mathcal{O}_{W_\sigma}(W_\sigma)$ be the image of $w_{\{\sigma(n),\ldots,\sigma(i+1)\}}$. Equivalently, we can use the restricted original data $((\mathscr{L}_I|_{W_\sigma}, w_I|_{W_\sigma})_I, (c_{ij}|_{W_\sigma})_{ij})$ and the image of the respective product of the restricted $w_I$'s under $c_{\sigma(i+1),\sigma(i)}|_{W_\sigma}$.

From these functions $w_1^\sigma, \ldots, w_{n-1}^\sigma$ we construct an $n$-pointed chain over $W_\sigma$ embedded in the projective space $\mathbb{P}^n_{W_\sigma}$ with homogeneous coordinates $y_0, \ldots, y_n$. Let $C_\sigma$ be the subscheme of $\mathbb{P}^n_{W_\sigma}$ defined by the equations $y_i y_{j+1} = w_{i+1}^\sigma \cdots w_j^\sigma y_{i+1} y_j$ for $0 \leq i < j < n$ (cf. construction 3.6), the sections $s^\sigma_{\sigma(i)}$ defined by the additional equation $y_{i-1} = y_i \neq 0$, and let $s_-^\sigma, s_+^\sigma$ be the sections $(1:0:\ldots:0), (0:\ldots:0:1)$.

These $n$-pointed chains $(C_\sigma \to W_\sigma, s_-^\sigma, s_+^\sigma, s_1^\sigma, \ldots, s_n^\sigma)$ can be glued to an $n$-pointed chain $(C \to Y, s_-, s_+, s_1, \ldots, s_n)$ over $Y$.

**Proposition 4.13.** *Construction 4.12 is valid and defines an isomorphism between the functor $C_{\Sigma(A_{n-1})}$ and the moduli functor of $n$-pointed chains of projective lines mapping the universal $\Sigma(A_{n-1})$-collection $((\mathcal{O}_{X(A_{n-1})}(D_I), w_I)_I, (c_{ij})_{ij})$ to the universal $n$-pointed chain $(X(A_n) \to X(A_{n-1}), s_-, s_+, s_1, \dots, s_n)$.*

*Proof.* Given $\Sigma(A_{n-1})$-data over a scheme $Y$, it is easy to show that construction 4.12 locally over the open subschemes $W_\sigma \subseteq Y$ defines $n$-pointed chains of projective lines (compare also to construction 3.6).

We show that, applying the isomorphism of functors $\overline{L}_n \to F_{A_{n-1}}$ to these objects over $W_\sigma$, we obtain $A_{n-1}$-data which coincide with the data we get by applying the functor $C_{\Sigma(A_{n-1})} \to F_{A_{n-1}}$ to the restricted data. According to [BB11a, Section 3.3] we extract $A_{n-1}$-data from an $n$-pointed chain $(C_\sigma \to W_\sigma, s_-^\sigma, s_+^\sigma, s_1^\sigma, \dots, s_n^\sigma)$ via projections to $\mathbb{P}^1_{W_\sigma}$ such that $s_-^\sigma, s_+^\sigma$ become the $(1:0), (0:1)$-section and a given section $s_i^\sigma$ becomes the section $(1{:}1)$. In the present case for $i = 1, \dots, n$ the morphism determined by the rational functions $1, y_i/y_{i-1}$ restricted to the component of $C_\sigma$ containing $s^\sigma_{\sigma(i)}$ after contracting the other components transforms the sections $s_-^\sigma, s_+^\sigma, s^\sigma_{\sigma(i)}$ into the $(1:0), (0:1), (1:1)$-sections. For $n = 1, \dots, n-1$ the section $s^\sigma_{\sigma(i+1)}$ becomes the section $(w_i^\sigma{:}1)$, and this gives $(t_{-\beta_{\sigma(i),\sigma(i+1)}} : t_{\beta_{\sigma(i),\sigma(i+1)}}) = (w_i^\sigma{:}1)$ which coincides with the data obtained via proposition 4.1.

Thus, the chains over the open subschemes $W_\sigma$ can be glued to an $n$-pointed chain over the scheme $Y$ and construction 4.12 defines a morphism of functors $C_{\Sigma(A_{n-1})} \to \overline{L}_n$, such that its composition with $\overline{L}_n \to F_{A_{n-1}}$ coincides with the isomorphism of functors $C_{\Sigma(A_{n-1})} \to F_{A_{n-1}}$ defined in proposition 4.1. Since the other morphisms of functors are isomorphisms and map the given universal objects to the given universal objects, this is also true for $C_{\Sigma(A_{n-1})} \to \overline{L}_n$. □

**Theorem 4.14.** *The morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$ that arises by forgetting the labels of the $n$ sections is given by the following $\Upsilon(A_{n-1})$-collection on $\overline{L}_n = X(A_{n-1})$: for $i = 1, \dots, n-1$ let $\mathscr{L}_{\varrho_i} = \mathcal{O}_{\overline{L}_n}(C_i)$ and $\mathscr{L}_{\tau_i} = \mathcal{O}_{\overline{L}_n}(D_i)$, let*

$$c_i: \ \mathscr{L}_{\tau_i}\mathscr{L}_{\varrho_{i-1}}\mathscr{L}_{\varrho_i}^{\otimes(-2)}\mathscr{L}_{\varrho_{i+1}} = \mathcal{O}_{\overline{L}_n}(D_i - C_{i-1} + 2C_i - C_{i+1}) \ \to \ \mathcal{O}_{\overline{L}_n}$$

*be given by multiplication by the rational function (8), and let the sections $a_i, b_i$ be defined as the images of the 1-sections under the inclusions $\mathcal{O}_{\overline{L}_n} \to \mathscr{L}_{\varrho_i}$, $\mathcal{O}_{\overline{L}_n} \to \mathscr{L}_{\tau_i}$.*

*Proof.* The data defined form an $\Upsilon(A_{n-1})$-collection, nondegeneracy follows from $C_i \cap D_i = \emptyset$, see lemma 4.10.

We use the covering by $W_\sigma$, $\sigma \in S_n$ (see remark 4.11). We have an isomorphism $((\mathscr{L}_{\varrho_i}, a_i)_i, (\mathscr{L}_{\tau_i}, b_i)_i, (c_i)_i)|_{W_\sigma} \to ((\mathcal{O}_{W_\sigma}, a_i^\sigma)_i, (\mathcal{O}_{W_\sigma}, b_i^\sigma)_i, (id)_i)$ of $\Upsilon(A_{n-1})$-collections on $W_\sigma$ consisting of isomorphisms $\mathscr{L}_{\tau_i}|_{W_\sigma} \to \mathcal{O}_{W_\sigma}$ given by multiplication with $x_{\sigma(i+1)}/x_{\sigma(i)}$ (compare to construction 4.12) and $\mathscr{L}_{\varrho_i}|_{W_\sigma} \to \mathcal{O}_{W_\sigma}$ by multiplication with $(\sum_{|I|=i} x_I)/x_{\{\sigma(1),\dots,\sigma(i)\}}$.

We show that the degree-$n$-pointed chain constructed from these data coincides with the degree-$n$-pointed chain that arises by forgetting the labels of the universal $n$-pointed chain coming from the universal $\Sigma(A_{n-1})$-collection by proposition 4.13.

Applying construction 3.6 to these data, we get a chain of projective lines $C \subset \mathbb{P}^n_{W_\sigma}$ defined by the functions $b_i^\sigma = x_{\sigma(i+1)}/x_{\sigma(i)}$ and a subscheme $S \subset C$ finite of degree $n$ over $W_\sigma$ defined by the functions $a_i^\sigma = (\sum_{|I|=i} x_I)/x_{\{\sigma(1),\ldots,\sigma(i)\}}$.

Applying construction 4.12 to the universal $\Sigma(A_{n-1})$-collection on $X(A_{n-1})$, locally over $W_\sigma \subset X(A_{n-1})$ again we get the chain of projective lines $C \subset \mathbb{P}^n_{W_\sigma}$ defined by the functions $b_i^\sigma = x_{\sigma(i+1)}/x_{\sigma(i)}$. The $n$ sections are

$$s_{\sigma(i)} = (\ \ldots\ : \frac{x^2_{\sigma(i)}}{x_{\sigma(i-2)}x_{\sigma(i-1)}} : \frac{x_{\sigma(i)}}{x_{\sigma(i-1)}} : 1 : 1 : \frac{x_{\sigma(i+1)}}{x_{\sigma(i)}} : \frac{x_{\sigma(i+1)}x_{\sigma(i+2)}}{x^2_{\sigma(i)}} :\ \ldots\ )$$

in terms the coordinates $y_0, \ldots, y_n$ of $\mathbb{P}^n_{W_\sigma}$, that is, we have $y_{i-1}(s_{\sigma(i)}) = y_i(s_{\sigma(i)})$ which we may set to 1, and then $y_k(s_{\sigma(i)}) = x^{i-k}_{\sigma(i)} \frac{x_{\sigma(1)}\cdots x_{\sigma(k)}}{x_{\sigma(1)}\cdots x_{\sigma(i)}}$. The sections are contained in the hyperplane defined by $\sum_{k=0}^n (-)^k a_k^\sigma y_k = 0$ (set $a_0^\sigma = a_n^\sigma = 1$):

$$\begin{aligned}
\textstyle\sum_{k=0}^n (-)^k a_k^\sigma y_k(s_{\sigma(i)}) &= \frac{x^i_{\sigma(i)}}{x_{\sigma(1)}\cdots x_{\sigma(i)}} \textstyle\sum_{k=0}^n (-)^k \sum_{|I|=k} \frac{x_I\, x_{\sigma(1)}\cdots x_{\sigma(k)}}{x_{\{\sigma(1),\ldots,\sigma(k)\}} x^k_{\sigma(i)}} \\
&= \frac{x^i_{\sigma(i)}}{x_{\sigma(1)}\cdots x_{\sigma(i)}} \textstyle\sum_{k=0}^n (-)^k \Big( \sum_{\substack{|I|=k \\ \sigma(i)\notin I}} \frac{x_I\, x_{\sigma(1)}\cdots x_{\sigma(k)}}{x_{\{\sigma(1),\ldots,\sigma(k)\}} x^k_{\sigma(i)}} + \sum_{\substack{|I|=k \\ \sigma(i)\in I}} \frac{x_I\, x_{\sigma(1)}\cdots x_{\sigma(k)}}{x_{\{\sigma(1),\ldots,\sigma(k)\}} x^k_{\sigma(i)}} \Big) \\
&= \frac{x^i_{\sigma(i)}}{x_{\sigma(1)}\cdots x_{\sigma(i)}} \textstyle\sum_{k=0}^n \Big( (-)^k \sum\limits_{\substack{|I|=k \\ \sigma(i)\notin I}} \frac{x_I\, x_{\sigma(1)}\cdots x_{\sigma(k)}}{x_{\{\sigma(1),\ldots,\sigma(k)\}} x^k_{\sigma(i)}} - (-)^{k-1} \sum\limits_{\substack{|I|=k-1 \\ \sigma(i)\notin I}} \frac{x_I\, x_{\sigma(1)}\cdots x_{\sigma(k-1)}}{x_{\{\sigma(1),\ldots,\sigma(k-1)\}} x^{k-1}_{\sigma(i)}} \Big) \\
&= 0
\end{aligned}$$

The relative effective divisors $\sum_i s_i$ and $S$ in $C$ over $W_\sigma$ coincide since they coincide over the open dense subscheme of $\overline{L}_n$ parametrising chains with distinct sections. $\square$

**Remark 4.15.** The results of this section imply a construction of a morphism of fibred categories from the functor of $\Sigma(A_{n-1})$-collections, considering the $S_n$-operation on this functor, to the category of $\boldsymbol{\Upsilon}(A_{n-1})$-collections such that the diagram

$$\begin{array}{ccc}
C_{\Sigma(A_{n-1})} & \overset{\sim}{\longleftrightarrow} & \overline{L}_n \\
\downarrow & & \downarrow \\
\mathcal{C}_{\boldsymbol{\Upsilon}(A_{n-1})} & \overset{\sim}{\longleftrightarrow} & \overline{\mathcal{L}}_n
\end{array}$$

commutes.

**Example 4.16.** In the case $n = 2$ the functor $F_{A_1} \cong C_{\Sigma(A_1)} \to \mathcal{C}_{\boldsymbol{\Upsilon}(A_1)}$ maps an object $(\mathscr{L}_{\{\pm\beta_{12}\}}, \{t_{\beta_{12}}, t_{-\beta_{12}}\})$ of $F_{A_1}$ over a scheme $Y$ to the object

$$\begin{aligned}
&(\mathcal{O}_Y(C_1), a_1), (\mathcal{O}_Y(D_1), b_1), c_1 \colon \mathcal{O}_Y(D_1) \otimes \mathcal{O}_Y(C_1)^{\otimes\text{-}2} \xrightarrow{\sim} \mathcal{O}_Y) \cong \\
&((\mathscr{L}_{\{\pm\beta_{12}\}}, t_{\beta_{12}} + t_{-\beta_{12}}), (\mathscr{L}^{\otimes 2}_{\{\pm\beta_{12}\}}, t_{\beta_{12}} t_{-\beta_{12}}), \mathscr{L}^{\otimes 2}_{\{\pm\beta_{12}\}} \otimes \mathscr{L}^{\otimes\text{-}2}_{\{\pm\beta_{12}\}} \xrightarrow{\sim} \mathcal{O}_Y)
\end{aligned}$$

of $\mathcal{C}_{\boldsymbol{\Upsilon}(A_1)}$, cf. example 1.8.

## 5. Pointed chains with involution and Cartan matrices of type $B$ and $C$

As a natural variation of $\overline{\mathcal{L}}_n$ we consider moduli stacks $\overline{\mathcal{L}}_n^{\pm}$ of stable degree-$2n$-pointed chains of projective lines with an involution.

**Definition 5.1.** We define the fibred category $\overline{\mathcal{L}}_n^{\pm}$ of stable degree-$2n$-pointed chains of projective lines with involution. An object over a scheme $Y$ is a collection $(C \to Y, I, s_-, s_+, S)$, where $(C \to Y, s_-, s_+, S)$ is a stable degree-$2n$-pointed chain of projective lines over $Y$ (definition 1.1), $I$ an automorphism of $C$ over $Y$ such that $I^2 = id_C$ and $I(s_-) = s_+$, and $S$ is invariant under $I$. Morphisms between objects are morphisms of degree-$2n$-pointed chains which commute with the involution $I$.

As in the case of $\overline{\mathcal{L}}_n$, see proposition 1.4, the fibred category $\overline{\mathcal{L}}_n^{\pm}$ is a stack in the fpqc topology with representable finite diagonal.

Considering degree-$2n$-pointed chains of projective lines with involution as degree-$2n$-pointed chains defines a morphism of stacks $\overline{\mathcal{L}}_n^{\pm} \to \overline{\mathcal{L}}_{2n}$ which makes $\overline{\mathcal{L}}_n^{\pm}$ a subcategory of $\overline{\mathcal{L}}_{2n}$ but in general not a substack, because a stable degree-$2n$-pointed chain may have automorphisms not commuting with an additional involution.

The moduli stack $\overline{\mathcal{L}}_n^{\pm}$ decomposes, unless we are working in characteristic 2, into two components $\overline{\mathcal{L}}_n^{\pm} = \overline{\mathcal{L}}_{n,+}^{\pm} \cup \overline{\mathcal{L}}_{n,-}^{\pm}$, where the component $\overline{\mathcal{L}}_{n,+}^{\pm}$ parametrises isomorphism classes of stable degree-$2n$-pointed chains with involution $(C, I, s_-, s_+, S)$ such that the degree of $S$ in each of the fixed points under the involution is even. We first consider this main component $\overline{\mathcal{L}}_{n,+}^{\pm}$.

The component $\overline{\mathcal{L}}_{n,+}^{\pm}$ is related to the moduli space $\overline{L}_n^{\pm} \cong X(C_n)$ of $2n$-pointed chains with involution defined in [BB11b, Section 6]. There is a morphism $\overline{L}_n^{\pm} \to \overline{\mathcal{L}}_{n,+}^{\pm}$ forgetting the labels of the sections. This morphism is equivariant with respect to the natural action of the Weyl group $W(C_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ on $\overline{L}_n^{\pm}$, the coarse moduli space of $\overline{\mathcal{L}}_{n,+}^{\pm}$ is $\overline{L}_n^{\pm}/W(C_n)$. Similar as in proposition 1.5 one can show that the morphism $\overline{L}_n^{\pm} \to \overline{\mathcal{L}}_{n,+}^{\pm}$ is faithfully flat and finite of degree $|W(C_n)| = 2^n n!$.

These morphisms together with the morphisms $\overline{L}_{2n} \to \overline{\mathcal{L}}_{2n}$ (see section 1) and $\overline{L}_n^{\pm} \to \overline{L}_{2n}$ (see [BB11b, Rem. 6.16]) form a commutative diagram

$$\begin{array}{ccc} \overline{L}_n^{\pm} & \longrightarrow & \overline{L}_{2n} \\ \downarrow & & \downarrow \\ \overline{\mathcal{L}}_{n,+}^{\pm} & \longrightarrow & \overline{\mathcal{L}}_{2n} \end{array}$$

where $\overline{L}_n^{\pm}$ is a component of the fibred product.

The stack $\overline{\mathcal{L}}_{n,+}^{\pm}$ compactifies the space of finite subschemes of degree $2n$ in $\mathbb{P}^1 \setminus \{0, \infty\}$ which are invariant under the involution and of even degree in each of the fixed points of the involution. Equivalently, this is the space of polynomials $\sum_{i=0}^{2n} a_i' y^i$ of degree $2n$ with the symmetry $a_{2n-i}' = a_i'$ in the coefficients, up to change

of the variable by multiplication by $-1$. These polynomials can contain $y-1$ and $y+1$ only with even multiplicity. After dividing by the coefficient $a'_{2n} = a'_0$, we have a polynomial of the form

$$y^{-n} + a_{n-1}y^{-n+1} + \ldots + a_1 y^{-1} + a_0 + a_1 y + \ldots + a_{n-1}y^{n-1} + y^n$$

determined by the isomorphism class up to multiplication of $y$ with $-1$ (together with multiplication of the whole expression by $(-1)^n$).

In general, embedding a chain $(C, I, s_-, s_+, S)$ into the projective space $\mathbb{P}^{2n} = \mathbb{P}(H^0(C, \mathcal{O}_C(S)))$, the image of $C$ is given by equations arising from the $2 \times 2$ minors of a matrix of the form (decompose into several matrices if some of the $b_i$ are zero, cf. remark 3.3; symbol $\sqrt{b_0}$ introduced for symmetry reasons)

$$\begin{pmatrix} \cdots & y_{\text{-}2} & y_{\text{-}1} & \sqrt{b_0}y_0 & \sqrt{b_0}b_1y_1 & \cdots \\ \cdots & \sqrt{b_0}b_1y_{\text{-}1} & \sqrt{b_0}y_0 & y_1 & y_2 & \cdots \end{pmatrix}$$

where $y_{\text{-}n}, \ldots, y_0, \ldots, y_n$ is a basis of $H^0(C, \mathcal{O}_C(S))$ defined similar as in proposition 3.2, 3.4 and such that the involution maps $y_{\text{-}i} \leftrightarrow y_i$. The sections $s_-, s_+$ become the sections $(1:0:\ldots:0), (0:\ldots:0:1)$ and the subscheme $S \subset C \subset \mathbb{P}^{2n}$ is determined by an equation

$$y_{\text{-}n} + a_{n-1}y_{\text{-}(n-1)} + \ldots + a_1y_{\text{-}1} + a_0y_0 + a_1y_1 + \ldots + a_{n-1}y_{n-1} + y_n.$$

For an algebraically closed field $K$ a $K$-valued point of $\overline{\mathcal{L}}^{\pm}_{n,+}$ corresponds to a collection $(a_{n-1}, \ldots, a_0, b_{n-1}, \ldots, b_0) \in K^{2n}$ up to the equivalence

$$(a_{n-1}, \ldots, a_0, b_{n-1}, \ldots, b_0) \sim (\kappa_{n-1}a_{n-1}, \ldots, \kappa_0 a_0, \lambda_{n-1}b_{n-1}, \ldots, \lambda_0 b_0)$$

with $(\kappa_{n-1}, \ldots, \kappa_0, \lambda_{n-1}, \ldots, \lambda_0) \in (K^*)^{2n}$ satisfying

$$\lambda_{n-1} = \kappa_{n-1}^2/\kappa_{n-2},\ \lambda_{n-2} = \kappa_{n-2}^2/(\kappa_{n-3}\kappa_{n-1}),\ \ldots,\ \lambda_1 = \kappa_1^2/(\kappa_0\kappa_2),\ \lambda_0 = \kappa_0^2/\kappa_1^2.$$

This gives rise to a toric orbifold whose exact sequence of tori

$$1 \longrightarrow G \cong (\mathbb{G}_m)^n \longrightarrow (\mathbb{G}_m)^{2n} \longrightarrow T_N \cong (\mathbb{G}_m)^n \longrightarrow 1$$

corresponds to the exact sequence of lattices

$$0 \longrightarrow M \cong \mathbb{Z}^n \xrightarrow{\left(\begin{smallmatrix} \text{-}C \\ I_n \end{smallmatrix}\right)} \mathbb{Z}^{2n} \xrightarrow{(\begin{smallmatrix} I_n & C \end{smallmatrix})} \mathbb{Z}^n \longrightarrow 0$$

where $C = C(C_n)^\top$ is the transpose of the Cartan matrix

$$C(C_n) = \begin{pmatrix} 2 & \text{-}1 & 0 & \cdots & \cdots & 0 \\ \text{-}1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & \text{-}1 & 0 \\ \vdots & & \ddots & \text{-}1 & 2 & \text{-}1 \\ 0 & \cdots & \cdots & 0 & \text{-}2 & 2 \end{pmatrix}$$

of the root system $C_n$.

**Definition 5.2.** We define the toric orbifold $\mathcal{Y}(C_n)$ associated to the Cartan matrix of the root system $C_n$ in terms of the stacky fan $\mathbf{\Upsilon}(C_n) = (N, \Upsilon(C_n), \beta)$, where $N = \mathbb{Z}^n$ and the linear map $\beta\colon \mathbb{Z}^{2n} \to N$ is given by the $n{\times}2n$ matrix $(\text{-}C(C_n)\ I_n)$. The fan $\Upsilon(C_n)$ consists of the $2n$ one-dimensional cones $\varrho_{n-1}, \dots, \varrho_0, \tau_{n-1}, \dots, \tau_0$ generated by the columns of the matrix $(\text{-}C(C_n)\ I_n)$. A subset of one-dimensional cones generates a higher dimensional cone if it does not contain one of the sets $\{\varrho_0, \tau_0\}, \dots, \{\varrho_{n-1}, \tau_{n-1}\}$. This defines a fan containing $2^n$ $n$-dimensional cones $\sigma_I$ generated by sets $\{\varrho_i\}_{i\notin I} \cup \{\tau_i\}_{i\in I}$ for subsets $I \subseteq \{0, \dots, n-1\}$.

The functor of $\mathbf{\Upsilon}(C_n)$-collections $\mathcal{C}_{\mathbf{\Upsilon}(C_n)} \cong \mathcal{Y}(C_n)$ has objects of the form $((\mathscr{L}_{\varrho_i}, a_i)_{i=0,\dots,n-1}, (\mathscr{L}_{\tau_i}, b_i)_{i=0,\dots,n-1}, (c_i)_{i=0,\dots,n-1})$ over a scheme $Y$, where the $c_i$ are isomorphisms of line bundles on $Y$

$$c_{n-1}\colon \mathscr{L}_{\tau_{n-1}} \otimes \mathscr{L}_{\varrho_{n-1}}^{\otimes\text{-}2} \otimes \mathscr{L}_{\varrho_{n-2}} \to \mathcal{O}_Y,\ c_{n-2}\colon \mathscr{L}_{\tau_{n-2}} \otimes \mathscr{L}_{\varrho_{n-1}} \otimes \mathscr{L}_{\varrho_{n-2}}^{\otimes\text{-}2} \otimes \mathscr{L}_{\varrho_{n-3}} \to \mathcal{O}_Y,$$
$$\dots\dots\dots,\ c_1\colon \mathscr{L}_{\tau_1} \otimes \mathscr{L}_{\varrho_2} \otimes \mathscr{L}_{\varrho_1}^{\otimes\text{-}2} \otimes \mathscr{L}_{\varrho_0} \to \mathcal{O}_Y,\ c_0\colon \mathscr{L}_{\tau_0} \otimes \mathscr{L}_{\varrho_1}^{\otimes 2} \otimes \mathscr{L}_{\varrho_0}^{\otimes\text{-}2} \to \mathcal{O}_Y.$$

We have a morphism of stacks $\mathcal{C}_{\mathbf{\Upsilon}(C_n)} \to \mathcal{C}_{\mathbf{\Upsilon}(A_{2n-1})}$ by considering the collection

$$\begin{aligned}((\mathscr{L}_{\varrho_{n-1}}, a_{n-1}), \dots, (\mathscr{L}_{\varrho_0}, a_0), \dots, (\mathscr{L}_{\varrho_{n-1}}, a_{n-1}), (\mathscr{L}_{\tau_{n-1}}, b_{n-1}), \dots \\ \dots, (\mathscr{L}_{\tau_0}, b_0), \dots, (\mathscr{L}_{\tau_{n-1}}, b_{n-1}), c_{n-1}, \dots, c_0, \dots, c_{n-1}),\end{aligned} \tag{9}$$

built out of an $\mathbf{\Upsilon}(C_n)$-collection, as an $\mathbf{\Upsilon}(A_{2n-1})$-collection. This morphism can be described by the map of fans $\mathbf{\Upsilon}(C_n) \to \mathbf{\Upsilon}(A_{2n-1})$ mapping $e'_{n-1} \mapsto e_{2n-1} + e_1, \dots, e'_1 \mapsto e_{n+1} + e_{n-1}, e'_0 \mapsto e_n$, where $e'_{n-1}, \dots, e'_0$ are the generators of $\tau_{n-1}, \dots, \tau_0$ of $\mathbf{\Upsilon}(C_n)$ and $e_{2n-1}, \dots, e_1$ are those of $\tau_{2n-1}, \dots, \tau_1$ of $\mathbf{\Upsilon}(A_{2n-1})$. It corresponds to a toric morphism $\mathcal{Y}(C_n) \to \mathcal{Y}(A_{2n-1})$ making $\mathcal{Y}(C_n)$ a subcategory of $\mathcal{Y}(A_{2n-1})$.

**Theorem 5.3.** *There is an isomorphism of stacks* $\overline{\mathcal{L}}^{\pm}_{n,+} \cong \mathcal{Y}(C_n)$.

*Proof.* Applying construction 3.5 to a degree-$2n$-pointed chain with involution, it is possible to choose $y_0, \dots, y_{2n}$ such that the involution maps $y_i \leftrightarrow y_{2n-i}$, and we obtain a $\mathbf{\Upsilon}(A_{2n-1})$-collection of the form (9). Applying construction 3.6 to a $\mathbf{\Upsilon}(A_{2n-1})$-collection of the form (9), making symmetric choices, we can introduce an involution on the resulting degree-$2n$-pointed chain by $y_i \leftrightarrow y_{2n-i}$. A $\mathbf{\Upsilon}(A_{2n-1})$-collection of the form (9) is equivalent to a $\mathbf{\Upsilon}(C_n)$-collection, and further, morphisms of the corresponding degree-$2n$-chains with involution that commute with the involution are equivalent to morphisms of $\mathbf{\Upsilon}(C_n)$-collections. □

The case of the other component $\overline{\mathcal{L}}^{\pm}_{n,-}$ is very similar. The stack $\overline{\mathcal{L}}^{\pm}_{n,-}$ parametrises isomorphism classes of stable degree-$2n$-pointed chains with involution $(C, I, s_-, s_+, S)$ such that the degree of $S$ in each of the fixed points of the involution is odd if there are two fixed points, and positive if there is only one fixed point. It is related to the moduli stack $\mathcal{X}(C_{n-1})$ defined in [BB11b, Section 6]: there is a morphism $\mathcal{X}(C_{n-1}) \to \overline{\mathcal{L}}^{\pm}_{n,-}$ determined by forgetting the labels of the sections and adding the fixed point subscheme of the involution as a subscheme of degree 2 to the $2n-2$ sections.

The stack $\overline{\mathcal{L}}^{\pm}_{n,-}$ compactifies the stack of finite subschemes $S$ of degree $2n$ in $\mathbb{P}^1 \setminus \{0, \infty\}$ invariant under the involution such that $S$ has odd degree in $(1{:}1)$ and

(1 : -1) (positive degree in (1 : 1) = (1 : -1) in characteristic 2). Equivalently we may consider polynomials $\sum_{i=0}^{2n} a'_i y^i$ of degree $2n$ with the symmetry $a'_{2n-i} = -a'_i$ in the coefficients and $a'_n = 0$. We may represent each isomorphism class by an expression of the form

$$-y^{-n} - a_{n-1}y^{-n+1} - \ldots - a_1 y^{-1} + 0 + a_1 y + \ldots + a_{n-1}y^{n-1} + y^n$$

determined up to multiplication of $y$ with -1 (together with multiplication of the whole expression by $(-1)^n$). It has a factor $(y-1)(y^{-1}+1)$, occurring with odd multiplicity in characteristic $\neq 2$.

In general, similar as in the case of $\overline{\mathcal{L}}^{\pm}_{n,+}$, a not necessarily irreducible chain can be naturally embedded into $\mathbb{P}^{2n} = \mathbb{P}(H^0(C, \mathcal{O}_C(S)))$ and described by equations arising from the $2 \times 2$ minors of a matrix of the form

$$\begin{pmatrix} \cdots & y_{-2} & y_{-1} & y_0 & b_1 y_1 & \cdots \\ \cdots & b_1 y_{-1} & y_0 & y_1 & y_2 & \cdots \end{pmatrix}$$

and the subscheme $S \subset C \subset \mathbb{P}^{2n}$ is given by an equation

$$y_{-n} + a_{n-1}y_{-(n-1)} + \ldots + a_1 y_{-1} + a_1 y_1 + \ldots + a_{n-1}y_{n-1} + y_n.$$

Over an algebraically closed field $K$ a $K$-valued point of $\overline{\mathcal{L}}^{\pm}_{n,-}$ corresponds to a collection $(a_{n-1}, \ldots, a_1, b_{n-1}, \ldots, b_1) \in K^{2n-2}$ up to the equivalence

$$(a_{n-1}, \ldots, a_1, b_{n-1}, \ldots, b_1) \sim (\kappa_{n-1}a_{n-1}, \ldots, \kappa_1 a_1, \lambda_{n-1}b_{n-1}, \ldots, \lambda_1 b_1)$$

with $(\kappa_{n-1}, \ldots, \kappa_1, \lambda_{n-1}, \ldots, \lambda_1) \in (K^*)^{2n-2}$ satisfying

$$\lambda_{n-1} = \kappa_{n-1}^2/\kappa_{n-2},\ \lambda_{n-2} = \kappa_{n-2}^2/(\kappa_{n-3}\kappa_{n-1}),\ \ldots,\ \lambda_2 = \kappa_2^2/(\kappa_1\kappa_3),\ \lambda_1^2 = \kappa_1^2/\kappa_2^2.$$

We will see that this stack can be described by a toric stack that differs from $\mathcal{Y}(C_{n-1})$ by replacing the matrix $(\text{-}C(C_{n-1})\ I_{n-1})$ defining the map $\beta$ of the stacky fan $\boldsymbol{\Upsilon}(C_{n-1})$ by the matrix

$$\left(\begin{array}{c|cccc} & 1 & 0 & & \\ \text{-}C(C_{n-1}) & 0 & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ & & & 0 & 2 \end{array}\right).$$

In the case $n = 1$ we define it to be $B\mu_2$. This toric stack corresponds to the category of collections of the form $((\mathscr{L}_{\varrho_i}, a_i)_{i=1,\ldots,n-1}, (\mathscr{L}_{\tau_i}, b_i)_{i=1,\ldots,n-1}, (c_i)_{i=1,\ldots,n-1})$ over a scheme $Y$, where the $c_i$ are isomorphisms of line bundles

$$c_{n-1}\colon \mathscr{L}_{\tau_{n-1}} \otimes \mathscr{L}_{\varrho_{n-1}}^{\otimes\text{-}2} \otimes \mathscr{L}_{\varrho_{n-2}} \to \mathcal{O}_Y,\ c_{n-2}\colon \mathscr{L}_{\tau_{n-2}} \otimes \mathscr{L}_{\varrho_{n-1}} \otimes \mathscr{L}_{\varrho_{n-2}}^{\otimes\text{-}2} \otimes \mathscr{L}_{\varrho_{n-3}} \to \mathcal{O}_Y,$$
$$\ldots\ldots\ldots,\ c_2\colon \mathscr{L}_{\tau_2} \otimes \mathscr{L}_{\varrho_3} \otimes \mathscr{L}_{\varrho_2}^{\otimes\text{-}2} \otimes \mathscr{L}_{\varrho_1} \to \mathcal{O}_Y,\ c_1\colon \mathscr{L}_{\tau_1}^{\otimes 2} \otimes \mathscr{L}_{\varrho_2}^{\otimes 2} \otimes \mathscr{L}_{\varrho_1}^{\otimes\text{-}2} \to \mathcal{O}_Y.$$

**Proposition 5.4.** *The stack $\overline{\mathcal{L}}^{\pm}_{n,-}$ can be embedded into $\overline{\mathcal{L}}^{\pm}_{n,+}$ as the divisor $D_{\varrho_0}$ corresponding to the cone $\varrho_0$ and defined by $a_0 = 0$. It is isomorphic to the above toric stack.*

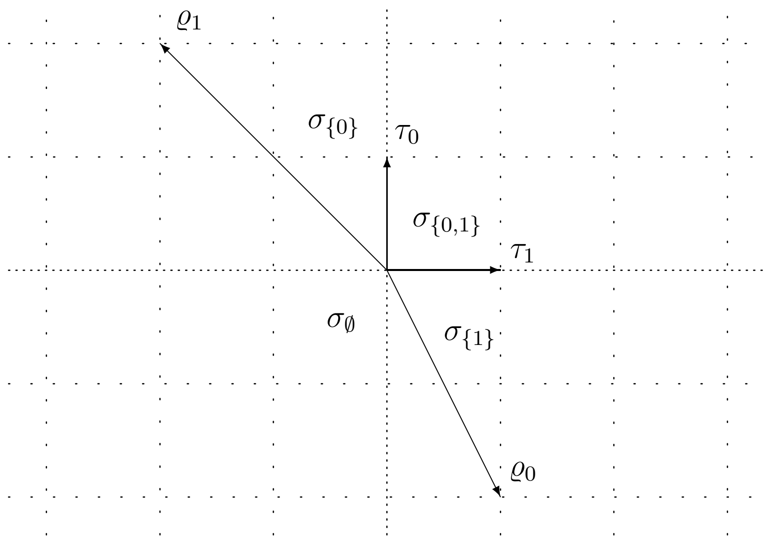
*Proof.* A chain with involution $(C \to Y, I, s_-, s_+)$ with a central component isomorphic to $\mathbb{P}^1_Y$ comes with the following relative effective divisors of degree 2: the invariant subschemes $S_I, S_{IJ}$ under the involutions $I, IJ$ of $\mathbb{P}^1_Y$, where $J$ is the involution that leaves the poles fixed and interchanges the two points $(1:1), (1:\text{-}1)$

of $S_I$. A degree-$2n$-pointed chain with involution $(C \to Y, I, s_-, s_+, S)$ is contained in $\overline{\mathcal{L}}^{\pm}_{n,-}$ if $S$ is of the form $S = S_I + S'$ and contained in the divisor $D_{\varrho_0}$ defined by $a_0 = 0$ if $S = S_{IJ} + S'$ for some relative effective divisor $S'$ that may contain $S_I$ only with even multiplicity. We have an isomorphism $\overline{\mathcal{L}}^{\pm}_{n,-} \cong D_{\varrho_0}$ mapping $(C \to Y, I, s_-, s_+, S_I + S') \leftrightarrow (C \to Y, I, s_-, s_+, S_{IJ} + S')$. The divisor $D_{\varrho_0}$ is the toric orbifold corresponding to the fan defined by the above matrix. □

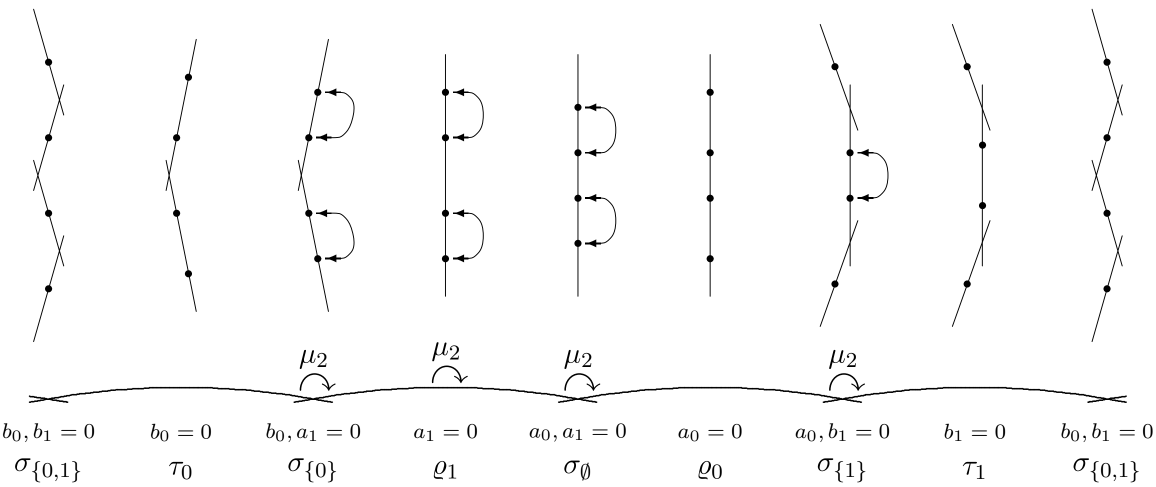
**Example 5.5.** The toric orbifold $\overline{\mathcal{L}}^{\pm}_{1,+}$ is isomorphic to the weighted projective line $\mathbb{P}(1,2) \cong \mathcal{Y}(C_1)$. Here the inclusion as subcategory $\overline{\mathcal{L}}^{\pm}_{1,+} \to \overline{\mathcal{L}}_2$ is an isomorphism of stacks as any degree-2-pointed chain is isomorphic to a symmetric object under an involution whose isomorphisms commute with the involution. So we have the same situation as in examples 1.8, 2.2, 3.8. The component $\overline{\mathcal{L}}^{\pm}_{1,-}$ is isomorphic to $B\mu_2$.

**Example 5.6.** The stacky fan of the toric orbifold $\overline{\mathcal{L}}^{\pm}_{2,+} \cong \mathcal{Y}(C_2)$ is given by the matrix $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix}$:

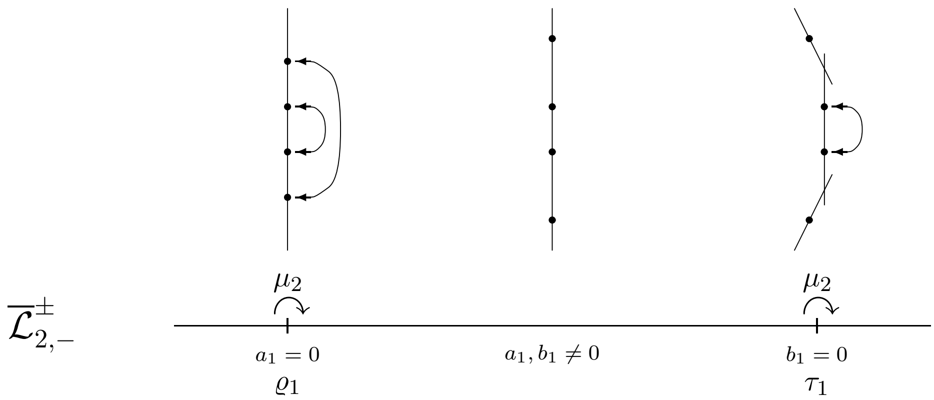
$\Upsilon(C_2)$




We picture the types of pointed chains over the torus invariant divisors of the moduli stack $\overline{\mathcal{L}}^{\pm}_{2,+}$.



The toric orbifold $\overline{\mathcal{L}}^{\pm}_{2,-} \cong \mathbb{P}^1/\mu_2$ corresponds to the stacky fan given by the matrix $\begin{pmatrix} -2 & 2 \end{pmatrix}$. We have the following types of pointed chains over $\overline{\mathcal{L}}^{\pm}_{2,-}$:




One also may consider chains with involution and a subscheme of odd degree.

**Definition 5.7.** Let the fibred category $\overline{\mathcal{L}}^{0,\pm}_n$ of stable degree-($2n$+1)-pointed chains of projective lines with involution be defined analogously to definition 5.1.

The fibred category $\overline{\mathcal{L}}^{0,\pm}_n$ is a stack in the fpqc topology with representable finite diagonal.

The moduli stack $\overline{\mathcal{L}}^{0,\pm}_n$ forms a subcategory of $\overline{\mathcal{L}}_{2n+1}$. It is related to the moduli space $\overline{L}^{0,\pm}_n \cong X(B_n)$ of ($2n$+1)-pointed chains with involution defined in [BB11b, Section 1]. We have a morphism $\overline{L}^{0,\pm}_n \to \overline{\mathcal{L}}^{0,\pm}_n$ forgetting the labels of the sections, which is equivariant with respect to the action of the Weyl group $W(B_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ on $\overline{L}^{0,\pm}_n$. The coarse moduli space of $\overline{\mathcal{L}}^{0,\pm}_n$ is $\overline{L}^{0,\pm}_n/W(B_n)$. As in the $C$-case the morphism $\overline{L}^{0,\pm}_n \to \overline{\mathcal{L}}^{0,\pm}_n$ is faithfully flat and finite of degree $|W(B_n)| = 2^n n!$, and we have a commutative diagram

$$\begin{array}{ccc} \overline{L}^{0,\pm}_n & \longrightarrow & \overline{L}_{2n+1} \\ \downarrow & & \downarrow \\ \overline{\mathcal{L}}^{0,\pm}_n & \longrightarrow & \overline{\mathcal{L}}_{2n+1} \end{array}$$

Embedding a degree-($2n+1$)-pointed chain with involution $(C, I, s_-, s_+, S)$ into the projective space $\mathbb{P}^{2n+1} = \mathbb{P}(H^0(\mathcal{O}_C(S)))$, the image of $C$ is given by equations arising from the $2 \times 2$ minors of a matrix of the form

$$\begin{pmatrix} \cdots & y_{\text{-}5/2} & y_{\text{-}3/2} & y_{\text{-}1/2} & b_1 y_{1/2} & b_1 b_2 y_{3/2} & \cdots \\ \cdots & b_1 b_2 y_{\text{-}3/2} & b_1 y_{\text{-}1/2} & y_{1/2} & y_{3/2} & y_{5/2} & \cdots \end{pmatrix}$$

and the subscheme $S$ by

$$y_{\text{-}(2n+1)/2} + a_n y_{\text{-}(2n-1)/2} + \ldots + a_1 y_{\text{-}1/2} + a_1 y_{1/2} + \ldots + a_n y_{(2n-1)/2} + y_{(2n+1)/2}.$$

where $y_{\text{-}(2n+1)/2}, \ldots, y_{\text{-}3/2}, y_{\text{-}1/2}, y_{1/2}, y_{3/2}, \ldots, y_{(2n+1)/2}$ is a basis of $H^0(\mathcal{O}_C(S))$ defined similar as in proposition 3.2, 3.4 and such that the involution maps $y_{\text{-}i/2} \leftrightarrow y_{i/2}$.

**Definition 5.8.** We define the toric orbifold $\mathcal{Y}(B_n)$ in terms of the stacky fan $\boldsymbol{\Upsilon}(B_n)$ as in definition 5.2 replacing the Cartan matrix $C(C_n)$ of the root system $C_n$ by the Cartan matrix $C(B_n)$ of the root system $B_n$.

It turns out that $\overline{\mathcal{L}}_n^{0,\pm}$ is not quite $\mathcal{Y}(B_n)$, but coincides with the underlying canonical toric stack $\mathcal{Y}(B_n)^{\text{can}}$ (as defined in [FMN10]). So instead of the Cartan matrix of the root system $B_n$ we have the matrix

$$\begin{pmatrix} 2 & \text{-}1 & 0 & \cdots & \cdots & 0 \\ \text{-}1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & \text{-}1 & 0 \\ \vdots & & \ddots & \text{-}1 & 2 & \text{-}1 \\ 0 & \cdots & \cdots & 0 & \text{-}1 & 1 \end{pmatrix}$$

where the rightmost column is half of the column of the Cartan matrix. The functor of $\boldsymbol{\Upsilon}(B_n)^{\text{can}}$-collections $\mathcal{C}_{\boldsymbol{\Upsilon}(B_n)^{\text{can}}} \cong \mathcal{Y}(B_n)^{\text{can}}$ has objects of the form $((\mathscr{L}_{\varrho_i}, a_i)_{i=1,\ldots,n}, (\mathscr{L}_{\tau_i}, b_i)_{i=1,\ldots,n}, (c_i)_{i=1,\ldots,n})$ over a scheme $Y$, where the $c_i$ are isomorphisms of line bundles

$$\begin{aligned} &c_n \colon \mathscr{L}_{\tau_n} \otimes \mathscr{L}_{\varrho_n}^{\otimes -2} \otimes \mathscr{L}_{\varrho_{n-1}} \to \mathcal{O}_Y,\ c_{n-1} \colon \mathscr{L}_{\tau_{n-1}} \otimes \mathscr{L}_{\varrho_n} \otimes \mathscr{L}_{\varrho_{n-1}}^{\otimes -2} \otimes \mathscr{L}_{\varrho_{n-2}} \to \mathcal{O}_Y, \\ &\ldots\ldots\ldots,\ c_2 \colon \mathscr{L}_{\tau_2} \otimes \mathscr{L}_{\varrho_3} \otimes \mathscr{L}_{\varrho_2}^{\otimes -2} \otimes \mathscr{L}_{\varrho_1} \to \mathcal{O}_Y,\ c_1 \colon \mathscr{L}_{\tau_1} \otimes \mathscr{L}_{\varrho_2} \otimes \mathscr{L}_{\varrho_1}^{\otimes -1} \to \mathcal{O}_Y. \end{aligned}$$

The inclusion as subcategory $\mathcal{Y}(B_n)^{\text{can}} \to \mathcal{Y}(A_{2n})$ can be described as $\mathcal{C}_{\boldsymbol{\Upsilon}(B_n)^{\text{can}}} \to \mathcal{C}_{\boldsymbol{\Upsilon}(A_{2n})}$ by considering the collection

$$\begin{gathered} ((\mathscr{L}_{\varrho_n}, a_n), \ldots, (\mathscr{L}_{\varrho_1}, a_1), (\mathscr{L}_{\varrho_1}, a_1), \ldots, (\mathscr{L}_{\varrho_n}, a_n), \\ (\mathscr{L}_{\tau_n}, b_n), \ldots, (\mathscr{L}_{\tau_1}, b_1), (\mathscr{L}_{\tau_1}, b_1), \ldots, (\mathscr{L}_{\tau_n}, b_n), c_n, \ldots, c_1, c_1, \ldots, c_n), \end{gathered}$$

formed out of an $\boldsymbol{\Upsilon}(B_n)^{\text{can}}$-collection, as an $\boldsymbol{\Upsilon}(A_{2n})$-collection.

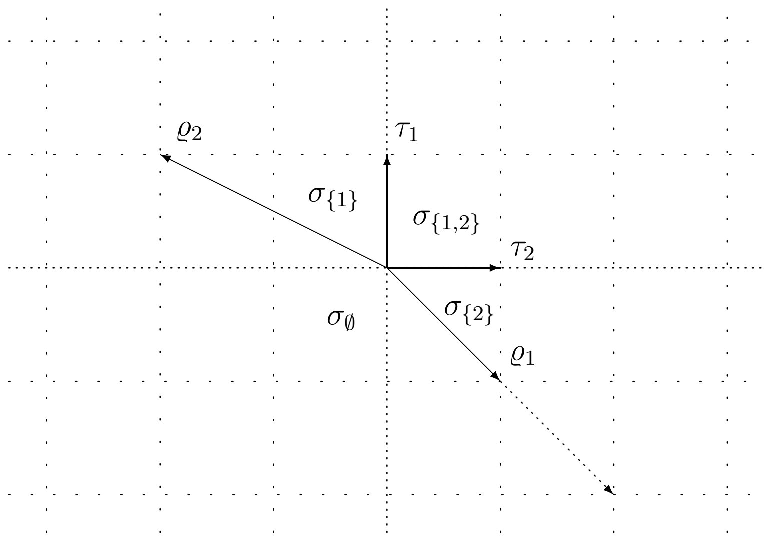
As in the case of degree-$2n$-pointed chains with involution one can prove:

**Theorem 5.9.** *There is an isomorphism of stacks* $\overline{\mathcal{L}}_n^{0,\pm} \cong \mathcal{Y}(B_n)^{can}$.

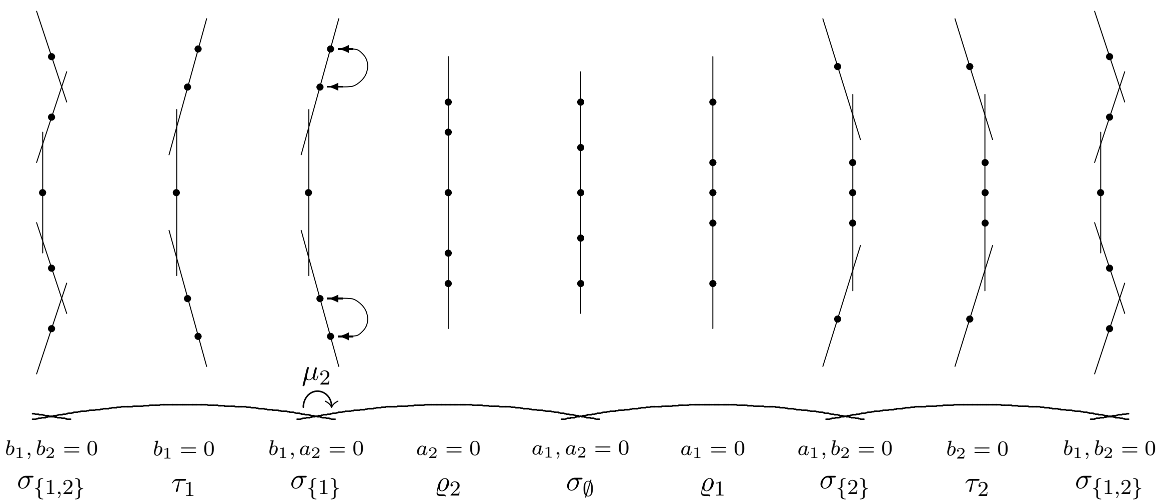
**Example 5.10.** In the case $n = 1$ we have a scheme $\overline{\mathcal{L}}_1^{0,\pm} \cong \mathcal{Y}(B_1)^{\mathrm{can}}$ isomorphic to the projective line $\mathbb{P}^1$.

**Example 5.11.** The toric orbifold $\overline{\mathcal{L}}_2^{0,\pm} \cong \mathcal{Y}(B_2)^{\mathrm{can}}$ is given by $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$. In the picture of the stacky fan $\boldsymbol{\Upsilon}(B_2)^{\mathrm{can}}$ the dotted arrow corresponds to the generator of the ray $\varrho_1$ determined by the stacky fan $\boldsymbol{\Upsilon}(B_2)$.

$\boldsymbol{\Upsilon}(B_2)^{\mathrm{can}}$




We picture the types of pointed chains over the torus invariant divisors of the moduli stack $\overline{\mathcal{L}}_2^{0,\pm}$.



## References


[AOV08] D. Abramovich, M. Olsson, A. Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier 58 (2008), 1057-1091, arXiv:math/0703310.

[BB11a] V. Batyrev, M. Blume, *The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces*, Tohoku Math. J. 63 (2011), arXiv:0911.3607.

[BB11b] V. Batyrev, M. Blume, *On generalisations of Losev-Manin moduli spaces for classical root systems*, Pure and Applied Mathematics Quarterly 7 (2011) (Special Issue: In memory of Eckart Viehweg), 1053–1084, arXiv:0912.2898.

[BCS05] L. Borisov, L. Chen, G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. 18 (2005), 193–215, arXiv:math/0309229.

[Co95a] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. 4 (1995), 17-50, arXiv:alg-geom/9210008.

[Co95b] D. Cox, *The functor of a smooth toric variety*, Tohoku Math. J. 47 (1995), 251–262, arXiv:alg-geom/9312001.

[EGA] A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique*, Publ. Math. IHES 4,8,11,17,20,24,28,32 (1960-1967).

[FMN10] B. Fantechi, E. Mann, F. Nironi, *Smooth toric Deligne-Mumford stacks*, J. reine angew. Math. 648 (2010), 201–244, arXiv:0708.1254.

[Gi] J. Giraud, *Cohomologie non abélienne*, Grundlehren math. Wiss. 179, Springer-Verlag, Berlin - Heidelberg - New York, 1971.

[Iw07] I. Iwanari, *Integral Chow rings of toric stacks*, arXiv:0705.3524.

[Ka93] M. Kapranov, *Chow quotients of Grassmannians I*, Adv. Soviet Math. 16 (1993), 29–110, arXiv:alg-geom/9210002.

[Kl05] S. Kleiman, *The Picard scheme*, in *Fundamental Algebraic Geometry*, Math. Surveys Monogr. 123, Amer. Math. Soc., Providence, RI, 2005, 237-321, arXiv:math/0504020.

[Kn83] F. Knudsen, *The projectivity of the moduli space of stable curves II: The stacks $M_{g,n}$*, Math. Scand. 52 (1983), 161–199.

[LM00] A. Losev, Yu. Manin, *New Moduli Spaces of Pointed Curves and Pencils of Flat Connections*, Michigan Math. J. 48 (2000), 443–472, arXiv:math/0001003.

[LMB] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Springer-Verlag, Berlin - Heidelberg - New York, 2000.

[Pe08] F. Perroni, *A note on toric Deligne-Mumford stacks*, Tohoku Math. J. 60 (2008), 441–458, arXiv:0705.3823.

[SGA4(3)] A. Grothendieck et al., *Séminaire de géométrie algébrique, Théorie des topos et cohomologie étale des schémas*, Tome 3, Lecture Notes in Mathematics 305, Springer-Verlag, Berlin - Heidelberg - New York, 1973.

[Vi05] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, in *Fundamental Algebraic Geometry*, Math. Surveys Monogr. 123, Amer. Math. Soc., Providence, RI, 2005, 1-104, arXiv:math/0412512.



Mathematisches Institut, Universität Münster,
Einsteinstrasse 62, 48149 Münster, Germany

*E-mail address*: mark.blume@uni-muenster.de